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Hamilton paths in vertex-transitive graphs of order $10p$

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ABSTRACT

It is shown that every connected vertex-transitive graph of order $10p$, $p \neq 7$ a prime, which is not isomorphic to a quasiprimitive graph arising from the action of $\mathrm{PSL}(2, k)$ on cosets of $\mathbb{Z}_k \rtimes \mathbb{Z}_{(k-1)/10}$, contains a Hamilton path.

1. Introductory remarks

In 1969, Lovász [28] asked if every finite, connected vertex-transitive graph has a Hamilton path, that is, a simple path going through all vertices of the graph. With the exception of K_2, only four connected vertex-transitive graphs that do not have a Hamilton cycle are known to exist. These four graphs are the Petersen graph, the Coxeter graph and the two graphs obtained from them by replacing each vertex by a triangle. The fact that none of these four graphs is a Cayley graph has led to a folklore conjecture that every connected Cayley graph with order greater than 2 has a Hamilton cycle.

Many articles directly and indirectly related to this subject have appeared in the literature (see [1–4,7,11,14,17–20,24,26,23,30–33,35,36,41,43,46,48–50] for some of the relevant references), affirming the existence of such paths and, in some cases, even Hamilton cycles. For example, it is known that connected vertex-transitive graphs of order kp, where $k \leq 5$, (except for the Petersen graph and the Coxeter graph), of order p^j, where $j \leq 4$, and of order $2p^2$, where p is a prime, contain a Hamilton cycle. It is also known that connected vertex-transitive graphs of order pq, where p and q are primes, admitting an imprimitive subgroup of automorphisms contain a Hamilton cycle. A Hamilton path is known to exist in connected vertex-transitive graphs of order $6p$. In addition, it is known that every connected vertex-transitive graph whose automorphism group contains a transitive subgroup with a cyclic commutator subgroup of prime-power order, with the exception of the Petersen graph, has a Hamilton cycle (this result was obtained with a generalization of the method used in [14,23,30]). We refer the reader to [25] for the current status of this problem.

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This paper deals with the existence of Hamilton paths in connected vertex-transitive graphs of order $10p$, where p is a prime. (Throughout this paper p will always denote a prime number.) The main object of this paper is to show that, with the exception of a certain family of graphs arising from the action of $\mathrm{PSL}(2, k)$ on cosets of $\mathbb{Z}_k \rtimes \mathbb{Z}_{(k-1)/10}$, every connected vertex-transitive graph of order $10p$, $p \neq 7$, contains a Hamilton path.

Theorem 1.1. *Let X be a connected vertex-transitive graph of order $10p$, where $p \neq 7$ is a prime, not isomorphic to a quasiprimitive graph arising from the action of $\mathrm{PSL}(2, k)$ on cosets of $\mathbb{Z}_k \rtimes \mathbb{Z}_{(k-1)/10}$. Then X contains a Hamilton path.*

The main tool in proving Theorem 1.1 is the so-called *lifting Hamilton cycles approach*, a frequently used approach for constructing Hamilton paths and cycles in vertex-transitive graphs. This approach is based on a quotienting/reduction with respect to an imprimitivity block system of the corresponding automorphism group or with respect to a suitable semiregular automorphism, preferably one of prime order. In particular, every vertex-transitive graph is either genuinely imprimitive, quasiprimitive or primitive. Following the method in [25] we divide our investigation depending on which of these three families the graph in question belongs to. There is no primitive graph of order $10p$ for $p > 19$. Also, there is no quasiprimitive graph of order $10p$ for $p > 31$ arising from a group action different from the action of $\mathrm{PSL}(2, k)$ on cosets of $\mathbb{Z}_k \rtimes \mathbb{Z}_{(k-1)/10}$. For $p \leq 31$ all primitive and quasiprimitive graphs of order $10p$ are known and the existence of Hamilton cycles in such graphs (with the exception of the truncation of the Petersen graph) is proved with the help of program package Magma [5]. In particular, we construct all relevant graphs and in each of them we either find a transitive group of automorphisms with a cyclic commutator subgroup of prime-power order (and thus the above mentioned result proved in [11] applies) or we find a semiregular automorphism of prime order such that the corresponding quotient graph contains such a Hamilton cycle that it can be, with the use of the lifting Hamilton cycle approach, lifted to a Hamilton cycle of the original graph. For the genuinely imprimitive graphs we use the lifting Hamilton cycle approach based on a quotienting/reduction with respect to an imprimitivity block system formed by the orbits of a minimal normal subgroup of a genuinely imprimitive group of automorphisms. In particular, the investigation depends on the size of the blocks in such imprimitivity block systems.

The paper is organized as follows. In Section 2 notions concerning this paper are introduced together with the notation and some auxiliary results that are needed in the subsequent sections. The rest of the paper is devoted to proving Theorem 1.1. The genuinely imprimitive graphs are considered in Section 3, the quasiprimitive graphs are considered in Section 4, and the primitive graphs are considered in Section 5. Finally, the results are combined in Section 6, where the Theorem 1.1 is proved.

2. Notation and preliminary results

Throughout this paper graphs are finite, simple and undirected, and groups are finite, unless specified otherwise. Furthermore, a *multigraph* is a generalization of a graph in which we allow multiedges and loops. Given a graph X we let $V(X)$ and $E(X)$ be the vertex set and the edge set of X, respectively. For adjacent vertices $u, v \in V(X)$ we write $u \sim v$ and denote the corresponding edge by uv. The complement of a graph X will be denoted by X^c. Let U and W be disjoint subsets of $V(X)$. The subgraph of X induced by U will be denoted by $X\langle U\rangle$. Similarly, we let $X[U, W]$ (in short $[U, W]$) denote the bipartite subgraph of X induced by the edges having one endvertex in U and the other endvertex in W.

Given a transitive group G acting on a set V, we say that a partition $\mathcal{B}$ of V is *G-invariant* if the elements of G permute the parts, that is, *blocks* of $\mathcal{B}$, setwise. If the trivial partitions $\{V\}$ and $\{\{v\} \mid v \in V\}$ are the only G-invariant partitions of V, then G is said to be *primitive*, and is said to be *imprimitive* otherwise. In the latter case we shall refer to a corresponding G-invariant partition as a *complete imprimitivity block system*, in short an *imprimitivity block system*, of G.

A graph X is said to be *vertex-transitive* if its automorphism group, denoted by $\mathrm{Aut}(X)$, acts transitively on $V(X)$. A vertex-transitive graph for which each transitive subgroup of its automorphism group is primitive is called a *primitive graph*. Otherwise it is called an *imprimitive graph*. If X is

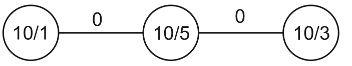

Fig. 1. The levi graph given in Frucht's notation with respect to a $(3, 10)$-semiregular automorphism.

imprimitive with an imprimitivity block system which is formed by the orbits of a proper normal subgroup of some transitive subgroup $G \leq \mathrm{Aut}(X)$, then the graph X is said to be *genuinely imprimitive*. If X is imprimitive, but there exists no transitive subgroup $G \leq \mathrm{Aut}(X)$ having a nontransitive normal subgroup, then X is said to be *quasiprimitive*. Note that if $\mathcal{B}$ is an imprimitivity block system of some vertex-transitive graph, then any two blocks $B, B' \in \mathcal{B}$ induce isomorphic vertex-transitive subgraphs.

Imprimitive vertex-transitive graphs of order $2p$, p a prime, were described in [29]. Among other things it was proved there that, provided a vertex-transitive graph X of order $2p$ admits an imprimitive group G (with blocks of size p or 2), one can always find an imprimitive subgroup of G which has blocks of size p. In particular, the following result is proved in [29] and will be used later.

Proposition 2.1. *Let X be a vertex-transitive graph of order $2p$, p a prime. If $G \leq \mathrm{Aut}(X)$ is an imprimitive subgroup of $\mathrm{Aut}(X)$ on X with blocks of size 2, then there exists an imprimitive subgroup H of G with blocks of size p.*

Given a graph X and a partition $\mathcal{P}$ of its vertex set we let the *quotient graph corresponding to* $\mathcal{P}$ be the graph $X_{\mathcal{P}}$ whose vertex set equals $\mathcal{P}$ with $A, B \in \mathcal{P}$ adjacent if there exist vertices $a \in A$ and $b \in B$, such that $a \sim b$ in X.

2.1. Semiregularity

Let $m \geq 1$ and $n \geq 2$ be integers. An automorphism of a graph is called (m, n)*-semiregular* if it has m orbits of length n and no other orbit. Now let X be a graph admitting an (m, n)-semiregular automorphism ρ and denote the set of the orbits of ρ by $\mathcal{S}$ (When we discuss the orbits of an individual permutation ρ, we mean the orbits of the cyclic subgroup $\langle\rho\rangle$ generated by ρ.). Let $S, S' \in \mathcal{S}$. We let $d(S)$ and $d(S, S')$ denote the valency of $X\langle S\rangle$ and $X[S, S']$, respectively. (Clearly, the graph $X[S, S']$ is regular.) We let the *quotient multigraph corresponding to* ρ be the multigraph X_ρ whose vertex set is $\mathcal{S}$ and in which $S, S' \in \mathcal{S}$ are joined by $d(S, S')$ edges. Observe that $\mathcal{S}$ is a partition of $V(X)$, so we can also consider the quotient graph $X_{\mathcal{S}}$ which is precisely the underlying graph of X_ρ.

In the subsequent sections some of the graphs will be represented in Frucht's notation [15]. For the sake of completeness we include the definition. Let X be a connected graph of order mn admitting an (m, n)-semiregular automorphism ρ. Let $\mathcal{S} = \{S_i \mid i \in \mathbb{Z}_m\}$ be the set of orbits of ρ. Denote the vertices of X by v_i^j, where $i \in \mathbb{Z}_m$ and $j \in \mathbb{Z}_n$, in such a way that $S_i = \{v_i^j \mid j \in \mathbb{Z}_n\}$ with $v_i^j = (v_i^0)^{\rho^j}$. Then X may be represented by the notation of Frucht [15] emphasizing the m orbits of ρ in the following way. The m orbits of ρ are represented by m circles. The symbol n/R, where $R \subseteq \mathbb{Z}_n \setminus \{0\}$, inside a circle corresponding to the orbit S_i indicates that for each $j \in \mathbb{Z}_n$, the vertex v_i^j is adjacent to all the vertices v_i^{j+r}, where $r \in R$. When $X\langle S_i\rangle$ is an independent set of vertices we simply write n inside its circle. Finally, an arrow pointing from the circle representing the orbit S_i to the circle representing the orbit S_k, $k \neq i$, labeled by the set $T \subseteq \mathbb{Z}_n$ indicates that for each $j \in \mathbb{Z}_n$, the vertex $v_i^j \in S_i$ is adjacent to all the vertices v_k^{j+t}, where $t \in T$. When the label is 0, the arrow on the line may be omitted. An example illustrating this notation is given in Fig. 1.

A graph X admitting an (m, n)-semiregular automorphism is completely determined by the so-called *symbol*. However, we define it here only for graphs admitting a $(10, p)$-semiregular automorphism. Let ρ be a $(10, p)$-semiregular automorphism and let S_i, $i \in \mathbb{Z}_{10}$, be its orbits. Choose $s_i \in S_i$ and define the following subsets of $\mathbb{Z}_p$. For $i, j \in \mathbb{Z}_{10}$, we let $R_{i,j} = \{r \in \mathbb{Z}_p \mid s_i \sim s_j^{\rho^r}\}$. Note that $R_{j,i} = -R_{i,j}$. It is clear that the collection of all $R_{i,j}$ completely determines X. The 10×10-matrix $\mathcal{M}_\rho(X) = (R_{i,j})_{i,j}$, whose (i, j)-th entry is the set $R_{i,j}$, is the *symbol* of X relative to $(\rho, s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9)$. The symbols will be used in Sections 4 and 5 to give relevant quasiprimitive and primitive graphs of order $10p$, p a prime.

The following proposition which is a generalization of [29, Theorem 3.4] is given in [26, Lemma 2.1].

Proposition 2.2. *Let X be a vertex-transitive graph of order mp, where p is a prime and $m < p$, and let $G \le \mathrm{Aut}(X)$ be a transitive subgroup of automorphisms of X. Then there exists some (m, p)-semiregular automorphism ρ of X such that $\rho \in G$.*

We wrap up this subsection with two results about imprimitive graphs of certain degrees which will be useful later on. The first result is a reformulation of [26, Lemma 2.1], and the second result may be deduced from [9, Lemma 2].

Proposition 2.3. *Let X be a G-imprimitive graph of order mq, q a prime, with a G-invariant partition $\mathcal{B}$ and let $H \le G$ have m orbits of length q. Let S be an orbit of H and let $B \in \mathcal{B}$ be such that $B \cap S \neq \emptyset$. Then one of the following holds:*

(i) $|B \cap S| = 1$, *in which case* $|B \cap S'| = 1$ *for every orbit* S' *of* H *which meets* B, *or*
(ii) $B \cap S = S$, *in which case* q *divides* $|B|$.

Proposition 2.4. *Let X be a vertex-transitive graph of order mq, q a prime, let G be an imprimitive subgroup of automorphisms of X and let N be a normal subgroup of G with orbits of length q. Then X has an (m, q)-semiregular automorphism whose orbits coincide with the orbits of N.*

2.2. Existence of Hamilton cycles/paths in particular graphs

A path of a graph X which meets each of the vertices of X is called a *Hamilton path* of X. A *Hamilton cycle* is defined in a similar way. A graph X is *Hamiltonian* if it possesses a Hamilton cycle. The following classical result, due to Jackson [22], giving a sufficient condition for the existence of Hamilton cycles in 2-connected regular graphs will be used throughout this paper. (Note that every connected vertex-transitive graph is 2-connected.)

Proposition 2.5 (*[22, Theorem 6]*)**.** *Every 2-connected regular graph of order n and valency at least $n/3$ contains a Hamilton cycle.*

A graph is *Hamilton-connected* if for every pair of vertices there is a Hamilton path between the two vertices, and it is *edge-Hamiltonian* if each of its edges is contained in some Hamilton cycle. By the following proposition Cayley graphs on abelian groups are edge-Hamiltonian graphs.

Proposition 2.6 (*[8, Theorem 6]*)**.** *Let X be a connected Cayley graph on an abelian group of order at least three. Then each edge of X is contained in some Hamilton cycle of X.*

The following three results about the existence of Hamilton cycles in particular vertex-transitive graphs will be used in the proofs throughout this paper.

Proposition 2.7 (*[1]*)**.** *Let X be a connected vertex-transitive graph of order $2p$, p is a prime. Then X is the Petersen graph or X is Hamiltonian.*

A detailed description of connected vertex-transitive graphs of order qp, q and p primes, whose automorphism groups contain imprimitive subgroups is given in [37,38]. It was proved in [34] that with the exception of the Petersen graph every such graph has a Hamilton cycle. For $q = 5$ every connected vertex-transitive graph of order qp with a primitive automorphism group containing no imprimitive subgroups arises from one of the primitive groups of degree qp without imprimitive subgroups given in [38, Theorem 2.1], and that they are Hamiltonian was proved in [13]. Therefore, the following proposition holds.

Proposition 2.8. *Let X be a connected vertex-transitive graph of order $5p$, p a prime. Then X is the Petersen graph or X is Hamiltonian.*

Proposition 2.9 (*[11, Theorem 1.1]*)**.** *Let X be a connected vertex-transitive graph of order at least 3. If there is a transitive group G of automorphisms of X such that the commutator subgroup of G is cyclic of prime-power order, then X is the Petersen graph or X is Hamiltonian.*

We next introduce the following notion of a lift of a path in a graph with a semiregular automorphism. Let X be a graph that admits an (m, n)-semiregular automorphism ρ. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ be the set of orbits of ρ, let $X_{\mathcal{S}}$ be the corresponding quotient graph and let $\wp : X \to X_{\mathcal{S}}$ be the corresponding projection. Let $W = S_{i_1} S_{i_2} \cdots S_{i_k}$ be a path in $X_{\mathcal{S}}$. We let the *lift of the path* W be the set of all paths of X whose images under $\wp$ are W.

A frequently used approach to constructing Hamilton cycles in vertex-transitive graphs, which will also be used in this paper, is based on a quotienting/reduction with respect to a suitable semiregular automorphism, preferably one of prime order. Provided the quotient graph contains a Hamilton cycle it is sometimes possible to lift this cycle to construct a Hamilton cycle in the original graph, consistently spiraling through the corresponding orbits (see Example 2.11). Lifts of Hamilton cycles from quotient graphs which themselves have a Hamilton cycle are always possible, for example, where the quotienting is done relative to a semiregular automorphism of prime order and where in the quotient graph there are at least two adjacent orbits on the Hamilton cycle joined by a double edge. In this case one can always lift the Hamilton cycle from the quotient graph because the double edge gives us the possibility to conveniently "change direction" so as to get a walk in the quotient that lifts to a full cycle above. In particular, the following lemma is straightforward and is just a reformulation of [35, Lemma 5].

Proposition 2.10. *Let X be a graph admitting an (m, p)-semiregular automorphism ρ, where p is a prime. Let C be a cycle of length k in the quotient graph $X_{\mathcal{S}}$, where $\mathcal{S}$ is the set of orbits of ρ. Then, the lift of C either contains a cycle of length kp or it consists of p disjoint k-cycles. In the latter case we have $d(S, S') = 1$ for every edge SS' of C.*

Observe that for a given graph X admitting an (m, n)-semiregular automorphism ρ, the corresponding quotient graph X_ρ can be viewed as the graph whose vertices are circles in Frucht's notation of X with respect to ρ and edges are the edges between the circles. For an arc $e \in A(X_\rho)$ let $l(e)$ denote the label of the corresponding arc in Frucht's notation of X with respect to ρ. Similarly, for a walk W in X_ρ let $l(W)$ denote the sum of the labels of the arcs in Frucht's notation corresponding to the arcs belonging to the walk W. Throughout the paper the following observation is used frequently: If there exists a Hamilton cycle C of X_ρ such that $(l(C), n) = 1$ then X has a Hamilton cycle.

Example 2.11. The generalized orbital graph X arising from the action of PSL(2, 11) on cosets of D_6 with respect to a union of a self-paired suborbit of length 1 and a self-paired suborbit of length 3 contains a (10, 11)-semiregular automorphism ρ, and it can be nicely represented in Frucht's notation as shown in Fig. 2. Since $C = S_0S_1S_5S_7S_3S_8S_9S_6S_2S_4S_0$ is a Hamilton cycle in the quotient graph $X_{\mathcal{S}} = X_\rho$, where $\mathcal{S} = \{S_i \mid i \in \mathbb{Z}_{10}\}$ is the set of orbits of ρ, such that the sum of the labels of the arcs lying on C is equal to 9 (which is coprime to 11) this cycle can be lifted to a Hamilton cycle in the original graph X (see Fig. 2). This graph is one of the quasiprimitive graphs of order 110 arising from row 2 of Table 1, (see Section 4).

We end this subsection with a result about the existence of Hamilton paths in vertex-transitive graphs admitting a semiregular automorphism of prime order such that the corresponding quotient graph is of order congruent to 2 modulo 4 and is either isomorphic to a complete bipartite graph or a complete bipartite graph minus a matching.

Proposition 2.12. *Let X be a connected vertex-transitive graph of order $2qm$, where q is a prime and m is odd, admitting a $(2m, q)$-semiregular automorphism $\rho \in \mathrm{Aut}(X)$ and let $\mathcal{O}$ be the set of orbits of ρ. If $X_{\mathcal{O}} \in \{K_{m,m}, K_{m,m} - mK_2\}$, then X has a Hamilton path.*

Proof. Let $X_{\mathcal{O}} \in \{K_{m,m}, K_{m,m} - mK_2\}$ and let $\mathcal{O} = \{S_i, T_i \mid i \in \mathbb{Z}_m\}$ such that $\{S_i \mid i \in \mathbb{Z}_m\}$ and $\{T_i \mid i \in \mathbb{Z}_m\}$ are the two bipartite sets of $X_{\mathcal{O}}$. Since every edge of $X_{\mathcal{O}}$ belongs to some Hamilton cycle of $X_{\mathcal{O}}$, we may, by Proposition 2.10, assume that $X_\rho = X_{\mathcal{O}}$, that is, $d(S_i, T_j) = 1$ for every $i, j \in \mathbb{Z}_m$. Since X is regular it follows that $d(S) = d(S')$ for any two orbits $S, S' \in \mathcal{O}$. Moreover, since q is a prime either $d(S) = 0$ or $d(S) \geq 2$ is even. If $d(S) = 2$, then a Hamilton cycle of X exists by [3, Theorem 3.9], and if $d(S) \geq 4$, then [8, Theorem 4] implies that for every $S \in \mathcal{O}$ the subgraph $X\langle S\rangle$ is Hamilton-connected, and so a Hamilton cycle of X clearly exists. We may therefore assume that $d(S) = 0$ for

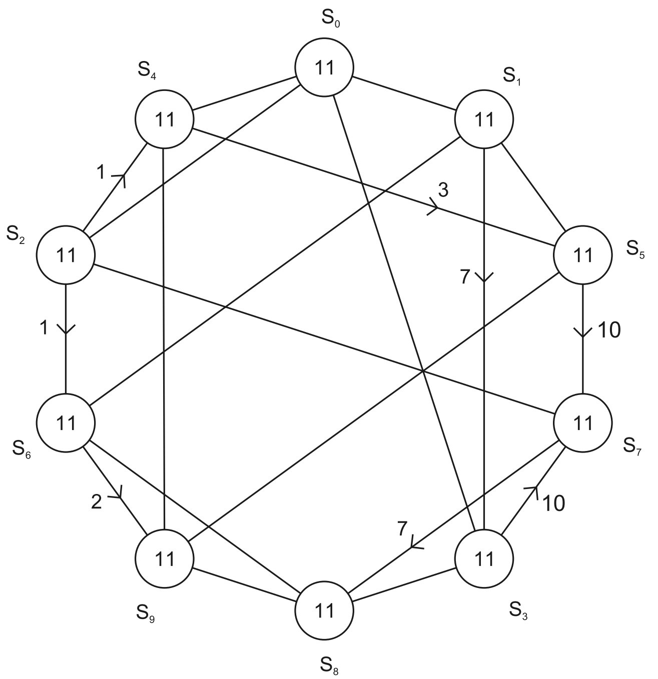

Fig. 2. A vertex-transitive graph arising from the action of PSL(2, 11) on cosets of D_6 given in Frucht's notation with respect to the (10, 11)-semiregular automorphism ρ where undirected lines carry label 0.

Table 1
Actions giving rise to quasiprimitive graphs of order $10p$.

Row	p	Action
1	7	A_7 on cosets of $\mathbb{Z}_3^2 \rtimes \mathbb{Z}_4$
2	11	PSL(2, 11) on cosets of D_6
3	11	PSL(2, 11) on cosets of $\mathbb{Z}_6$
4	31	PSL(3, 5) on cosets of $\mathbb{Z}_5^2 \rtimes (\mathbb{Z}_4 \cdot D_{12})$
5	31	PSL(3, 5) on cosets of $\mathbb{Z}_5^2 \rtimes (\mathbb{Z}_4 \cdot A_4)$
6	11	M_{11} on cosets of M_9
7	31	PSL(3, 5) on cosets of $\mathbb{Z}_5^2 \rtimes (\mathbb{Z}_{24} \cdot \mathbb{Z}_2)$
8	7	A_7 on cosets of $A_4 \times \mathbb{Z}_3$
9	7	PSL(4, 2) on cosets of $\mathbb{Z}_2^4 \rtimes (A_3 \times S_3)$
10	7	PSL(4, 2) on cosets of $\mathbb{Z}_2^4 \rtimes (A_3 \rtimes S_3)$
11	31	PSL(5, 2) on cosets of $\mathbb{Z}_2^6 \rtimes (A_3 \times \mathrm{PSL}(3, 2))$
12	13	PSL(2, 25) on cosets of PSL(2, 5)
13	11	M_{11} on cosets of $\mathbb{Z}_3^2 \rtimes \mathbb{Z}_8$
14	11	M_{11} on cosets of $\mathbb{Z}_3^2 \rtimes Q_8$
15	11	A_{11} on cosets of A_9
16	$\frac{k+1}{2}$	PSL(2, k) on cosets of $\mathbb{Z}_k \rtimes \mathbb{Z}_{(k-1)/10}$ where $5 \mid \frac{k-1}{2}$ and $k = s^m$

every $S \in \mathcal{O}$, that is, $X\langle S\rangle = qK_1$. We distinguish two different cases depending on whether $X_{\mathcal{O}} \cong K_{m,m}$ or $X_{\mathcal{O}} \cong K_{m,m} - mK_2$.

CASE 1. $X_{\mathcal{O}} \cong K_{m,m}$.

Then $S_i \sim T_j$ for every $i, j \in \mathbb{Z}_m$. If there exists a Hamilton cycle C of $X_\rho = X_{\mathcal{S}}$ such that $l(C) \neq 0$ then X clearly has a Hamilton cycle. Thus we may assume that no such Hamilton cycle exists in X_ρ. Also, if there exist two disjoint cycles C_1 and C_2 such that $V(X_\rho) = V(C_1) \cup V(C_2)$ and $l(C_1)$ and $l(C_2)$

are both different from 0, we have that C_1 and C_2 both lift to a single cycle in X since q is a prime, and consequently the connectedness of X implies that X has a Hamilton path. We may therefore assume that no such pair of cycles exists in X_ρ.

Since $S_0T_0S_1T_1\cdots S_{m-1}T_{m-1}S_0$ is a Hamilton cycle in X_ρ we may, without loss of generality, assume that $l(S_iT_i) = l(T_iS_{i+1}) = 0$ for every $i \in \mathbb{Z}_m$. Let $i \in \mathbb{Z}_m \setminus \{m-2, m-1\}$. Then

$$C_i = S_0T_iS_iT_{i-1}S_{i-1}\cdots S_1T_0S_0 \quad \text{and} \quad C_i' = S_{i+1}T_{m-1}S_{m-1}T_{m-2}\cdots S_{i+2}T_{i+1}S_{i+1}$$

are two disjoint cycles such that $V(X_\rho) = V(C_i) \cup V(C_i')$, $l(C_i) = l(S_0T_i)$ and $l(C_i') = l(S_{i+1}T_{m-1})$. If $l(C_i) = l(S_0T_i) \neq 0$ then, by the assumption made in the preceding paragraph, we have that $l(C_i') = l(S_{i+1}T_{m-1}) = 0$. Next, since

$$C_i'' = S_0T_{i+1}S_{i+2}T_{i+2}S_{i+3}\cdots S_{m-1}T_{m-1}S_{i+1}T_iS_iT_{i-1}\cdots S_1T_0S_0$$

is a Hamilton cycle of X_ρ with $l(C_i'') = l(S_0T_{i+1})$ we have $l(S_0T_{i+1}) = 0$. It follows that

$$D_i = S_0T_{i+1}S_{i+1}T_iS_iT_{i-1}\cdots S_1T_0S_{i+2}T_{i+2}S_{i+3}\cdots T_{m-1}S_0$$

is a Hamilton cycle of X_ρ with $l(D_i) = l(T_0S_{i+2})$, and thus $l(T_0S_{i+2}) = 0$. But then

$$S_0T_iS_iT_{i-1}S_{i-1}\cdots S_1T_0S_{i+2}T_{i+2}S_{i+3}\cdots T_{m-1}S_{i+1}T_{i+1}S_0$$

is a Hamilton cycle of X_ρ with a non-zero label and thus it lifts to a Hamilton cycle of X. It therefore follows that $l(S_0T_i) = 0$ for every $i \in \mathbb{Z}_m \setminus \{m-2\}$. Moreover, by replacing S_0 with an orbit $S_j, j \in \mathbb{Z}_m \setminus \{0\}$, in this argument, one can easily see that we have $l(S_jT_k) = 0$ whenever $|k-j| \neq m-2$. Further, since in the Hamilton cycle

$$C = S_0T_{m-2}S_{m-2}T_{m-3}S_{m-3}T_{m-1}S_{m-1}T_{m-4}S_{m-4}T_{m-5}\cdots S_1T_0S_0$$

of $X_\mathcal{O}$ the edge S_0T_{m-2} is the only edge of the form S_iT_{i+m-2}, we have that $l(C) = l(S_0T_{m-2})$, and thus $l(S_0T_{m-2}) = 0$. Since C^{ψ^j} is a Hamilton cycle of $X_\mathcal{O}$ and $l(C^{\psi^j}) = l((S_0T_{m-2})^{\psi^j}) = l(S_jT_{j+m-2})$, where $\psi = (S_0\ S_1 \cdots\ S_{m-1})(T_0\ T_1\ \cdots\ T_{m-1}) \in \mathrm{Aut}(X_\mathcal{O})$ and $j \in \mathbb{Z}_m$, we get that all the edges of $X_\mathcal{O}$ carry label 0. But then X is disconnected, a contradiction.

Case 2. $X_\mathcal{O} \cong K_{m,m} - mK_2$.

We can obtain $X_\mathcal{O}$ from the graph in Case 1 in such a way that we delete all the edges of the form $\{S_iT_{i+1} \mid i \in \mathbb{Z}_m\}$. Since none of the edges in the cycles, used in the proof of Case 1, is of the form $S_iT_{i+1}, i \in \mathbb{Z}_m$, we can apply the same argument as in Case 1 to show that X has a Hamilton path. $\square$

2.3. Group-theoretic results

A transitive group G acting on a set Ω is said to be *doubly transitive* if it acts transitively on the set of non-diagonal ordered pairs of points from Ω. Further, G is said to be *simply primitive* if it is primitive but not doubly transitive. The following result is due to Burnside [6].

Proposition 2.13. *Let G be a transitive group of prime degree p. Then either G is doubly transitive or G contains a normal Sylow p-subgroup.*

The following result on primitive groups of degree $2p$ may be deduced from [27].

Proposition 2.14. *A primitive group G of degree $2p$, p a prime, is one of the following:*

(i) $p = 5$, *and* $G = A_5$ *or* $G = S_5$;
(ii) $G = A_{2p}$ *or* $G = S_{2p}$;
(iii) $p = 11$ *and* $G = M_{22}$;
(iv) $p = \frac{1+q^{2^t}}{2}$, *and G is a subgroup of* $\mathrm{Aut}(\mathrm{PSL}(2,k))$ *containing* $\mathrm{PSL}(2,k)$, *where* $k = q^{2^t}$ *and q is an odd prime.*

Moreover, G is simply primitive in case (i) *and is doubly transitive in all other cases.*

The next result may be extracted from [12, Theorem 2.10].

Proposition 2.15. *Let G be a transitive permutation group of degree $10p, p \geq 5$ a prime, with an imprimitivity block system $\mathcal{B}$ formed by a (proper, intransitive) minimal normal subgroup N of G. Then N^B is simple for all blocks $B \in \mathcal{B}$.*

For the sake of completeness we state the following classical result which will be used throughout the paper.

Proposition 2.16 (*[47, Theorem 3.4]*). *Let p be a prime and let P be a Sylow p-subgroup of a permutation group G acting on a set Ω. Let $\omega \in \Omega$. If p^m divides the length of the G-orbit containing ω, then p^m also divides the length of the P-orbit containing ω.*

We wrap up this section by a result on imprimitive groups of degree $5p, p \geq 7$ a prime, which will be needed later on in this paper. In the proof of this result the following two propositions will be needed.

Proposition 2.17 (*[44]*). *Let G be a finite group and $H \leq G$. If $|G:H| = n$, then G/H_G is isomorphic to a subgroup of the symmetric group S_n, where H_G is the largest normal subgroup of G that is contained in H.*

Proposition 2.18 (*[21]*). *Let G be a non-abelian simple group, $H < G$ and $|G : H| = p^n$, where p is a prime and n is a positive integer. Then one of the following holds:*

(i) *$G = A_m$ and $H \cong A_{m-1}$, where $m = p^n$;*
(ii) *$G = \mathrm{PSL}(m, q)$, H is the stabilizer of a line or hyperplane, and $|G : H| = \frac{q^m-1}{q-1} = p^n$;*
(iii) *$G = \mathrm{PSL}(2, 11)$ and $H \cong A_5$;*
(iv) *$G = M_{23}$ and $H \cong M_{22}$, or $G = M_{11}$ and $H \cong M_{10}$;*
(v) *$G = U_4(2) \cong S_4(3)$ and H is the parabolic subgroup of index 27.*

Proposition 2.19. *Let G be a transitive non-abelian simple group of degree $5p, p \geq 7$ a prime, and let H be a maximal subgroup of G such that $G_\alpha < H < G$. Then G is quasiprimitive and one of the following holds:*

(i) *$G = \mathrm{PSL}(2, 11)$, $H = A_5$, $|G : H| = 11$ and $G_\alpha = A_4$;*
(ii) *$G = \mathrm{PSL}(m, q)$, H is the stabilizer of a line or hyperplane, m is a prime, q is a prime power and $|G : H| = \frac{q^m-1}{q-1} = p$.*

Proof. Let G be a transitive non-abelian simple group of degree $5p$, where $p \geq 7$ is a prime. Since the stabilizer G_α of a point α is not maximal the group G is imprimitive, and consequently, since it is a simple group, it is quasiprimitive.

By Praeger's classification of quasiprimitive groups [42], one can see that G is in class AS. Let H be a maximal subgroup of G such that $G_\alpha < H < G$. Since $|G : G_\alpha| = 5p$, we can conclude that either $|G : H| = 5$ or $|G : H| = p$. If $|G : H| = 5$, then $|H : G_\alpha| = p$, and since G is a non-abelian simple group, Proposition 2.17 implies that G is isomorphic to a subgroup of S_5. We can conclude that $G \cong A_5$ and thus $H \cong A_4$. But since $p \geq 7$, A_4 has no subgroup of index p, a contradiction. If, however, $|G : H| = p$, then $|H : G_\alpha| = 5$, and thus G is one of the groups listed in Proposition 2.18(i)–(iv).

Suppose that G is the group from Proposition 2.18(i). Then $G = A_p$ and $H \cong A_{p-1}$. For $p - 1 \leq 4$ the group $H \cong A_{p-1}$ has no subgroup of index 5. On the other hand, if $p - 1 \geq 5$, then $H \cong A_{p-1}$ is a simple group, and it has a subgroup of index 5 if and only if $p - 1 = 5$. But then $p = 6$ is not a prime, a contradiction. It follows that G is not a group from Proposition 2.18(i). Further, since M_{22} and M_{10} have no subgroup of index 5, G cannot be a group from Proposition 2.18(iv) either, and we can conclude that G is a group from Proposition 2.18(ii) or (iii). □

3. Genuinely imprimitive graphs

Throughout this section let X be a connected genuinely imprimitive graph of order $10p, p > 5$ a prime, admitting an imprimitive subgroup G of $\mathrm{Aut}(X)$ with a non-transitive minimal normal subgroup $N \lhd G$. Let the set of orbits of N be denoted by $\mathcal{B}$. Then $\mathcal{B}$ is a complete imprimitivity block system of G.

Lemmas 3.1 and 3.3–3.7, each of which covers a particular size of the blocks in $\mathcal{B}$, combined together imply that every connected genuinely imprimitive graph of order $10p$, $p > 7$ a prime, possesses a Hamilton path.

Lemma 3.1. *If the size of blocks in $\mathcal{B}$ is 2, then X has a Hamilton path.*

Proof. Since $X_{\mathcal{B}}$ is a connected vertex-transitive graph of order $5p$, by Proposition 2.8, it has a Hamilton cycle C. By Proposition 2.4, X has a $(5p, 2)$-semiregular automorphism whose set of orbits equals $\mathcal{B}$. Thus, by Proposition 2.10, either C lifts to a Hamilton cycle of X or it lifts to a disjoint union of two cycles of length $5p$. Since X is connected a Hamilton path exists in X. □

The following proposition about the graphs whose quotient graph with respect to $\mathcal{B}$ is isomorphic to the Petersen graph will be used in the proof of Lemma 3.3. The proposition is a direct generalization of [26, Lemma 3.2]. We omit the proof.

Proposition 3.2. *If the size of blocks in $\mathcal{B}$ is p and the quotient graph $X_{\mathcal{B}}$ is isomorphic to the Petersen graph, then X has a Hamilton path.*

In the proof of the next lemma we will be using the following notation. Let $C_n = (0, 1, \ldots, n-1)$ be an n-cycle. A graph C_n^+ is a graph with $V(C_n^+) = V(C_n)$ and $E(C_n^+) = E(C_n) \cup \{\{i, i+n/2\} \mid i \in \mathbb{Z}_n\}$ (clearly C_n^+ is well defined only for even integers n). A graph $C_n(k)$ is a graph with $V(C_n(k)) = V(C_n)$ and $E(C_n(k)) = E(C_n) \cup \{\{i, i+k\} \mid i \in \mathbb{Z}_n\}$. A graph $C_n(k, l)$ is a graph with $V(C_n(k, l)) = V(C_n)$ and $E(C_n(k, l)) = E(C_n) \cup \{\{i, i+k\}, \{i, i+l\} \mid i \in \mathbb{Z}_n\}$. Also, recall that the *direct product* $Y \times Z$ of graphs Y and Z is a graph with $V(Y \times Z) = V(Y) \times V(Z)$ and $E(Y \times Z) = \{\{(a, x), (b, y)\} \mid ab \in E(Y) \text{ and } xy \in E(Z)\}$.

Lemma 3.3. *If the size of blocks in $\mathcal{B}$ is p, then X has a Hamilton path.*

Proof. The quotient graph $X_{\mathcal{B}}$ is a connected vertex-transitive graph of order 10. By Proposition 3.2, we may assume that $X_{\mathcal{B}}$ is not isomorphic to the Petersen graph. By Proposition 2.3 the blocks of $\mathcal{B}$ coincide with the orbits of some $(10, p)$-semiregular automorphism $\rho \in G$ of X, which exists by Proposition 2.4. Let $\mathcal{S} = \{S_i \mid i \in \mathbb{Z}_{10}\}$ denote the set of orbits of ρ.

There exist eighteen connected vertex-transitive graphs of order 10 of which one is isomorphic to the Petersen graph, (see [40]). In particular, the quotient graph $X_{\mathcal{S}} = X_{\mathcal{B}}$ is isomorphic to one of the following seventeen graphs:

$$\begin{array}{llllll} C_{10}, & K_{5,5}, & C_{10}^+, & C_5 \times K_2, & (K_5 \times K_2)^c, & C_{10}(4), \\ C_{10}(2), & C_{10}(2,5), & C_{10}(4,5), & K_5 \times K_2, & GP(5,2)^c, & (C_5 \times K_2)^c, \\ (C_{10}^+)^c, & (2C_5)^c, & C_{10}^c, & (5K_2)^c, & K_{10}. & \end{array}$$

It is easy to see that in all these cases for any edge e of $X_{\mathcal{S}}$ there exists a Hamilton cycle of $X_{\mathcal{S}}$ containing e. Hence, by Proposition 2.10, we may assume that no multiedge exists in X_ρ. Since $X_{\mathcal{S}}$ is Hamiltonian we may label the orbits of ρ in such a way that $S_i \sim S_{i+1}$ for every $i \in \mathbb{Z}_{10}$. If there exists a Hamilton cycle of $X_{\mathcal{S}}$ whose lift contains a Hamilton cycle of X, there is nothing to prove. Therefore, we can assume that no such Hamilton cycle of $X_{\mathcal{S}}$ exists. Consequently, we may assume that $l(S_iS_{i+1}) = 0$ for every $i \in \mathbb{Z}_{10}$. Note also that we can assume that $X\langle S_i \rangle = pK_1$ for all $i \in \mathbb{Z}_{10}$. Namely, all the subgraphs $X\langle S_i \rangle$ are of the same valency, and if the subgraphs $X\langle S_i \rangle$ are of valency 2, then a Hamilton cycle of X exists by [3, Theorem 3.9], and if the subgraphs $X\langle S_i \rangle$ are of valency at least 4, then [8, Theorem 4] implies that each of $X\langle S_i \rangle$ is Hamilton-connected (that is, there exists a Hamilton path in $X\langle S_i \rangle$ connecting any two vertices), and so a Hamilton cycle of X clearly exists.

We distinguish seventeen different cases depending on which of the seventeen connected vertex-transitive graphs of order 10 the quotient graph $X_{\mathcal{S}}$ is isomorphic to.

If $X_{\mathcal{S}} \cong C_{10}$, then S_iS_{i+1}, where $i \in \mathbb{Z}_{10}$, are the only edges of $X_{\mathcal{S}}$, and so X is not connected, a contradiction.

If $X_{\mathcal{S}} \cong K_{5,5}$ or $X_{\mathcal{S}} \cong K_{5,5} - 5K_2 \cong (K_5 \times K_2)^c$, then by Proposition 2.12, X has a Hamilton path.

If $X_{\mathcal{S}} \cong C_{10}^{+}$, then in addition to the edges S_iS_{i+1}, also $S_0S_5, S_1S_6, S_2S_7, S_3S_8, S_4S_9 \in E(X_{\mathcal{S}})$. Let $r_0 = l(S_0S_5), r_1 = l(S_1S_6), r_2 = l(S_2S_7), r_3 = l(S_3S_8)$, and $r_4 = l(S_4S_9)$. Since

$$S_0S_5S_4S_3S_2S_1S_6S_7S_8S_9S_0, \qquad S_0S_1S_2S_3S_4S_9S_8S_7S_6S_5S_0,$$
$$S_0S_1S_6S_5S_4S_3S_2S_7S_8S_9S_0, \qquad S_0S_1S_2S_3S_8S_7S_6S_5S_4S_9S_0$$

are Hamilton cycles of $X_{\mathcal{S}}$, Proposition 2.10 implies that $r_0 + r_1 = 0, r_4 - r_0 = 0, r_1 + r_2 = 0$ and $r_3 + r_4 = 0$. It follows that $r_1 = r_3, r_0 = r_2 = r_4, r_1 = -r_4$. If $r_1 = 0$, then since p is odd it follows that $r_0 = r_1 = r_2 = r_3 = r_4 = 0$ and thus X is disconnected, a contradiction. If, however, $r_1 \neq 0$ then $r_0 \neq 0$ and since $S_0S_5S_6S_1S_2S_7S_8S_3S_4S_9S_0$ is a Hamilton cycle of $X_{\mathcal{S}}$, Proposition 2.10 implies that $r_0 - r_1 + r_2 - r_3 + r_4 = 0$, and so $3r_0 = 2r_1$. But then, since $r_1 = -r_0$, it follows that $5r_0 \equiv 0 \pmod p$, implying that $p = 5$, a contradiction.

If $X_{\mathcal{S}} \cong C_5 \times K_2$, then we may assume that in addition to the edges S_iS_{i+1}, also

$$S_1S_8, S_2S_7, S_3S_6, S_4S_0, S_5S_9 \in E(X_{\mathcal{S}}).$$

Let $r_0 = l(S_1S_8), r_1 = l(S_2S_7), r_2 = l(S_3S_6), r_3 = l(S_4S_0)$, and $r_4 = l(S_5S_9)$. Since

$$S_0S_1S_8S_9S_5S_6S_7S_2S_3S_4S_0, \qquad S_0S_1S_2S_7S_8S_9S_5S_6S_3S_4S_0,$$
$$S_0S_1S_2S_3S_6S_7S_8S_9S_5S_4S_0, \qquad S_0S_9S_5S_6S_7S_8S_1S_2S_3S_4S_0,$$

are Hamilton cycles of $X_{\mathcal{S}}$, Proposition 2.10 implies that $r_0 - r_4 - r_1 + r_3 = 0, r_1 - r_4 - r_2 + r_3 = 0, r_2 - r_4 + r_3 = 0$, and $-r_4 - r_0 + r_3 = 0$. Combining these equations we get that $r_1 = 2r_0 = 2r_2$ and $r_0 + r_2 = 0$. Since p is odd it follows from the first of these two equations that $r_0 = r_2$, and then we get from the second equation that $r_0 = r_2 = 0$. Hence $r_0 = r_1 = r_2 = 0$ and then from the above equations we get that $r_3 = r_4$. In view of the connectedness of X, we have that $r_3 = r_4 \neq 0$. But then $S_0S_1S_2S_3S_4S_0$ and $S_9S_8S_7S_6S_5S_9$ are disjoint 5-cycles in $X_{\mathcal{S}}$ that lift to $5p$-cycles in X. Since the vertex sets of the obtained $5p$-cycles are disjoint and X is connected, it follows that X has a Hamilton path.

The remaining twelve cases are dealt with in a similar manner. We leave the details to the reader. □

An n-bicirculant is a graph with a $(2, n)$-semiregular automorphism. Every n-bicirculant X can be represented by a triple of subsets of $\mathbb{Z}_n$ in the following way. Let φ be a $(2, n)$-semiregular automorphism of X, let U and W be the two orbits of $\langle\varphi\rangle$, and let $u \in U$ and $w \in W$. Let $S = \{s \in \mathbb{Z}_n \mid u \sim u\varphi^s\}$ be the symbol of the n-circulant induced on U and let R be the symbol of the n-circulant induced on W (relative to φ). Moreover, let $T = \{t \in \mathbb{Z}_n \mid u \sim w\varphi^t\}$. The ordered triple $[S, R, T]$ is the *symbol* of X relative to (φ, u, w). Note that $S = -S$ and $R = -R$ are symmetric, that is, inverse-closed subsets of $\mathbb{Z}_n$, and are independent of the particular choice of vertices u and w.

Lemma 3.4. *If $p > 7$ and the size of blocks in $\mathcal{B}$ is 5, then X has a Hamilton path.*

Proof. By Proposition 2.4 there exists a $(2p, 5)$-semiregular automorphism $\varphi \in G$ whose orbit set coincides with $\mathcal{B} = \{B_i \mid i \in \mathbb{Z}_{2p}\}$. Since $p > 5$ the graph $X_{\mathcal{B}}$ is a vertex-transitive graph of order $2p$, not isomorphic to the Petersen graph, and therefore, by Proposition 2.7, it contains a Hamilton cycle, say $C = B_0B_1 \cdots B_{2p-1}B_0$. In view of Proposition 2.10 we can assume that the lift of C consists of five disjoint $2p$-cycles. So $d(B_i, B_{i+1}) = 1$ for all $i \in \mathbb{Z}_{2p}$. Moreover, we can assume that $X\langle B\rangle = 5K_1$ for all $B \in \mathcal{B}$. Namely, if for some $B \in \mathcal{B}$ we have $X\langle B\rangle \cong Y$, where $Y \in \{C_5, K_5\}$, then since $\mathcal{B}$ is an imprimitivity block system we have $X\langle B'\rangle \cong Y$ for every block $B' \in \mathcal{B}$. Further, since $X_{\mathcal{B}}$ has a Hamilton cycle and since between any two adjacent blocks of $\mathcal{B}$ we have a perfect matching (as $\mathcal{B}$ is the set of orbits of a normal subgroup) one can easily see that X has a Hamilton path.

Let K be the kernel of the action of G on $\mathcal{B}$. Then depending on the (im)primitivity of the action of $\bar{G} = G/K$ on $X_{\mathcal{B}}$ three cases need to be considered. In particular, since $X_{\mathcal{B}}$ is of order $2p$ either $\bar{G}$ acts primitively on $X_{\mathcal{B}}$ or it acts imprimitively with blocks of size 2 or p. Following the notation given in [24] we denote these possible types of action of G by $(2p : 5)$, $(2 : p : 5)$, and $(p : 2 : 5)$, respectively.

Case 1. G is of type $(2p : 5)$.

In this case $\bar{G} = G/K$ acts primitively on $X_{\mathcal{B}}$. Since $p > 5$, Proposition 2.14, implies that $\bar{G}$ is doubly transitive on $X_{\mathcal{B}}$, and so $X_{\mathcal{B}}$ is isomorphic to K_{2p}. Since, by Proposition 2.6, every edge of K_{2p} is contained in some Hamilton cycle, by Proposition 2.10, we may assume that $X_{\mathcal{B}} = X_{\varphi} \cong K_{2p}$. Clearly, $X_{\mathcal{B}}$ can be viewed as the graph whose vertices are circles in Frucht's notation of X with respect to φ and edges are the edges between the circles. Observe that if there exists a Hamilton cycle C of $X_{\mathcal{B}}$ such that $l(C) \neq 0$, where $l(C)$ is the sum of the labels of the arcs belonging to C, then X has a Hamilton cycle. Therefore, we can assume that no such Hamilton cycle of $X_{\mathcal{B}}$ exists.

Let us relabel the vertices of $X_{\mathcal{B}} = X_{\varphi} \cong K_{2p}$ in such a way that $V(X_{\mathcal{B}}) = \{v_i \mid i \in \mathbb{Z}_{2p}\}$. Let $C = v_0v_1v_2v_3v_4 \cdots v_{2p-1}v_0$ be a Hamilton cycle of $X_{\mathcal{B}}$. Then $l(C) = 0$, and moreover we may assume that all the edges of $X_{\mathcal{B}}$ contained in C carry label 0. Further, since $v_0v_2v_1v_3v_4 \cdots v_{2p-1}v_0$ is a Hamilton cycle of $X_{\mathcal{B}}$, we have $l(v_0v_2) = t = -l(v_1v_3)$. In addition, observe that for every $i \in \mathbb{Z}_{2p}$

$$v_0v_1v_2 \cdots v_{i+1}v_iv_{i+2} \cdots v_{2p-1}v_0$$

is a Hamilton cycle of $X_{\mathcal{B}}$ and thus we have

$$l(v_iv_{i+2}) = \begin{cases} t & \text{if } i \text{ is even} \\ -t & \text{if } i \text{ is odd} \end{cases}$$

where $t \in \mathbb{Z}_5$.

If $p \cdot t \not\equiv 0 \pmod 5$, then p-cycles $v_0v_2 \cdots v_{2p-2}v_0$ and $v_1v_3 \cdots v_{2p-1}v_1$ of $X_{\mathcal{B}}$ lift to two disjoint $5p$-cycles in X. Since X is connected it is clear that X contains a Hamilton path in this case. We may therefore assume that $p \cdot t \equiv 0 \pmod 5$, that is, $t = l(v_iv_{i+2}) = 0$ for every $i \in \mathbb{Z}_{2p}$. Further,

$$C_i = v_0v_1 \cdots v_{i-1}v_{i+1}v_{i+2}v_iv_{i+3}v_{i+4} \cdots v_{2p-1}v_0, \quad i \in \mathbb{Z}_{2p}$$

is a Hamilton cycle of $X_{\mathcal{B}}$, and since $l(C_i) = l(v_iv_{i+3})$ we have that $l(v_iv_{i+3}) = 0$ for every $i \in \mathbb{Z}_{2p}$. Next,

$$C'_i = v_{i+4}v_iv_{i+3}v_{i+2}v_{i+1}v_{i-1}v_{i-2} \cdots v_{i+5}v_{i+4}, \quad i \in \mathbb{Z}_{2p}$$

is a Hamilton cycle of $X_{\mathcal{B}}$ with $l(C'_i) = l(v_iv_{i+4})$ and thus we have that also $l(v_iv_{i+4}) = 0$ for every $i \in \mathbb{Z}_{2p}$. Continuing inductively, we get that all the edges of $X_{\mathcal{B}}$ have label 0. But then X is disconnected, a contradiction.

CASE 2. $(2 : p : 5)$.

Then the action of $\bar{G}$ on $X_{\mathcal{B}}$ gives an imprimitivity block system with two blocks, say $\bar{\mathcal{C}}$ and $\bar{\mathcal{D}}$, of size p. Let $\mathcal{C}$ and $\mathcal{D}$ be the corresponding blocks of size $5p$ of G in X, and let H be the index 2 subgroup of G such that $\bar{H} = \bar{G}_{\mathcal{C}} = \bar{G}_{\mathcal{D}}$ is the corresponding block stabilizer. Therefore, for a block $B \in \mathcal{B}$ and a vertex $v \in B$, we have a sequence of groups $G_v \leq G_B \leq H \leq G$ giving the type $(2 : p : 5)$.

Now let $C = \{x_0, x_1, \ldots, x_4\} \in \mathcal{C}$ and $D = \{y_0, y_1, \ldots, y_4\} \in \mathcal{D}$. Since $p > 7$, Proposition 2.2 implies that there exists a $(10, p)$-semiregular automorphism $\pi \in G$ such that $\bar{\mathcal{C}}$ and $\bar{\mathcal{D}}$ are orbits of $\bar{\pi}$. Let $x_j^i = x_j^{\pi^i}$ and $y_j^i = y_j^{\pi^i}$, $i \in \mathbb{Z}_p$. Then we have that $\mathcal{C} = \{C_i \mid i \in \mathbb{Z}_p\}$ and $\mathcal{D} = \{D_i \mid i \in \mathbb{Z}_p\}$, where $C_i = \{x_j^i \mid j \in \mathbb{Z}_5\}$ and $D_i = \{y_j^i \mid j \in \mathbb{Z}_5\}$. Clearly, $\mathcal{B} = \{C_i, D_i \mid i \in \mathbb{Z}_p\}$.

SUBCASE 2.1. $\bar{H}^{\bar{\mathcal{C}}}$ is unfaithful.

Then $X_{\mathcal{B}}[\bar{\mathcal{C}}, \bar{\mathcal{D}}] = K_{p,p}$, and, by Propositions 2.10 and 2.12,we may assume that $X[C_i, D_j] \cong 5K_2$ for all $i, j \in \mathbb{Z}_p$, that is, $d(C_i, D_j) = 1$ for every $i, j \in \mathbb{Z}_p$. Moreover, all the edges C_iD_j in $X_{\mathcal{B}}$ carry label 0.

Recall that $X\langle B\rangle \cong 5K_1$ for every $B \in \mathcal{B}$. If $X\langle\mathcal{C}\rangle \cong X\langle\mathcal{D}\rangle \cong 5pK_1$, then the edge set of $X_{\mathcal{B}}$ is equal to the edge set of $X_{\mathcal{B}}[\bar{\mathcal{C}}, \bar{\mathcal{D}}]$, and thus X is disconnected, a contradiction. If, however, $X\langle\mathcal{C}\rangle \cong X\langle\mathcal{D}\rangle \not\cong 5pK_1$, and thus $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle \cong X_{\mathcal{B}}\langle\bar{\mathcal{D}}\rangle \not\cong pK_1$, then $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle$ is a connected p-circulant, that is, a Cayley graph on a cyclic group of order p. By Proposition 2.6, $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle$ is Hamiltonian and moreover, every edge of $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle$ belongs to a Hamilton cycle of $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle$. Let C_H be a particular Hamilton cycle of $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle$. If $l(C_H) \neq 0$, then C_H lifts to a Hamilton cycle of $X\langle\mathcal{C}\rangle$ (to a $5p$-cycle in X). Since $X\langle\mathcal{C}\rangle \cong X\langle\mathcal{D}\rangle$, also $X\langle\mathcal{D}\rangle$ contains a cycle of length $5p$, and the connectivity of X implies that X has a Hamilton path. Thus we may assume that $l(C_H) = 0$, and consequently that $d(C_i, C_j) = 1$ for any pair of adjacent orbits C_i and C_j in $X\langle\mathcal{C}\rangle$. Moreover, we may assume that every Hamilton cycle of $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle$, as well as every Hamilton cycle of $X_{\mathcal{B}}\langle\bar{\mathcal{D}}\rangle$, lifts to a disjoint union of five p-cycles.

Assume first that all arcs in $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle$ have label 0. Then all arcs belonging to C_H have label 0. Since X is connected there exists a Hamilton cycle D_H in $X_{\mathcal{B}}\langle\bar{\mathcal{D}}\rangle$ such that not all arcs belonging to D_H have label 0. This implies that there exists an arc e on D_H such that $l(D_H - e) \neq 0$. Let $e = uv$, and let $e' = u'v'$ be an arc of C_H. Since $X_{\mathcal{B}}[\bar{\mathcal{C}}, \bar{\mathcal{D}}] = K_{p,p}$ we have that $uu', vv' \in E(X)$, and consequently one can easily see that starting at the vertex u, following the cycle C_H till v, then using the edge vv', following the cycle D_H till u', and finally using the edge uu' gives a Hamilton cycle of $X_{\mathcal{B}}$ with non-zero label and thus X has a Hamilton cycle.

Assume now that not all arcs in $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle$ have label 0. However, we can, without loss of generality, assume that there exists an arc e in $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle$ with $l(e) = 0$. Moreover, without loss of generality we may assume that this arc e belongs to C_H. Then $l(C_H - e) = 0$. If all the arcs in $X_{\mathcal{B}}\langle\bar{\mathcal{D}}\rangle$ have label 0, then by applying the argument from the preceding paragraph to D_H, one can see that X has a Hamilton path. Thus we may assume that there exists an arc e' in $X_{\mathcal{B}}\langle\bar{\mathcal{D}}\rangle$ with a non-zero label. Since every edge of $X_{\mathcal{B}}\langle\bar{\mathcal{D}}\rangle$ is contained in a Hamilton cycle, there exists a Hamilton cycle of D_H containing e'. Since $l(D_H) = 0$ and $l(e') \neq 0$ it follows that $l(D_H - e') \neq 0$. Now we can construct a Hamilton cycle of X in a similar manner as in the preceding paragraph.

Subcase 2.2. $\bar{H}^{\bar{\mathcal{C}}}$ is faithful.

By Proposition 2.13, either $\bar{H}^{\bar{\mathcal{C}}}$ is solvable and contains a normal Sylow p-subgroup P, or $\bar{H}^{\bar{\mathcal{C}}}$ is non-solvable and doubly transitive.

Subsubcase 2.2.1. $\bar{H}^{\bar{\mathcal{C}}}$ is solvable.

Then a Sylow p-subgroup P of $\bar{H}^{\bar{\mathcal{C}}}$ is normal in $\bar{H}^{\bar{\mathcal{C}}}$ and thus $\bar{\pi} \in P$. Since $\bar{H}^{\bar{\mathcal{C}}}$ is faithful and solvable, $\bar{H}^{\bar{\mathcal{C}}} \cong \bar{H} \le A(1, p)$. Since $\bar{H}$ is primitive and $A(1, p)$ is of order $p(p-1)$, P is of order p, and so $\langle\bar{\pi}\rangle = P$. It follows that $\langle\bar{\pi}\rangle$ is a characteristic subgroup of $\bar{H}$, implying that $\langle\bar{\pi}\rangle$ is normal in $\bar{G}$, and finally that $\langle\pi\rangle$ is normal in G. But then X is a genuinely imprimitive graph with respect to an imprimitivity block system consisting of blocks of size p, and so, by Lemma 3.3, X has a Hamilton path.

Subsubcase 2.2.2. $\bar{H}^{\bar{\mathcal{C}}}$ is non-solvable.

Then $\bar{H}^{\bar{\mathcal{C}}}$ is doubly transitive, and thus either $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle \cong K_p$ or $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle \cong pK_1$. From the proof of [10, Theorem 3.2] we can see that either $X_{\mathcal{B}}[\bar{\mathcal{C}}, \bar{\mathcal{D}}] \cong K_{p,p}$ or the stabilizer of a vertex in $\bar{\mathcal{C}}$ has two orbits on $\bar{\mathcal{D}}$. Hence in all cases the bipartite subgraph $X_{\mathcal{B}}[\bar{\mathcal{C}}, \bar{\mathcal{D}}]$ is an arc-transitive graph of order $2p$. Since $X_{\mathcal{B}}$ is a connected vertex-transitive graph of order $2p$, $p > 7$ a prime, it contains a Hamilton cycle. If $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle \cong X_{\mathcal{B}}\langle\bar{\mathcal{D}}\rangle \cong pK_1$ then $X_{\mathcal{B}}[\bar{\mathcal{C}}, \bar{\mathcal{D}}] \cong X_{\mathcal{B}}$ is an arc-transitive graph, implying that every edge of $X_{\mathcal{B}}$ belongs to a Hamilton cycle of $X_{\mathcal{B}}$. Therefore we may assume that $X[B, B'] \cong 5K_1$ for any two adjacent blocks $B, B' \in \mathcal{B}$. If, however, $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle \cong X_{\mathcal{B}}\langle\bar{\mathcal{D}}\rangle \cong K_p$ then this graph is Hamilton-connected. Hence, if there exist two adjacent blocks $B, B' \in \mathcal{B}$ arising from an edge in $X_{\mathcal{B}}\langle\bar{\mathcal{C}}\rangle$ such that $X[B, B'] \ncong 5K_1$, then $X\langle\mathcal{C}\rangle$ has a Hamilton cycle, and thus clearly X has a Hamilton path. We may therefore assume that $X[B, B'] \cong 5K_1$ for any two adjacent blocks $B, B' \in \mathcal{B}$ in this case as well. Since $\mathcal{B}$ is the set of orbits of a normal subgroup N of $\mathrm{Aut}(X)$ and X is connected we can now easily see that $N \cong \mathbb{Z}_5$. An elementary exercise in group theory then shows that $N\langle\rho\rangle \cong \mathbb{Z}_{5p} = \langle\varphi\rangle$, where ρ is a $(10, p)$-semiregular automorphism of $\mathrm{Aut}(X)$ and φ is the generator of the cyclic group $N\langle\rho\rangle$, implying that X is a bicirculant. Let $[S, R, T]$ be one of its symbols corresponding to φ, such that $0 \in T$. If there exists some $t \in T$ for which $\langle t\rangle = \mathbb{Z}_{5p}$, where $\langle t\rangle$ is the additive subgroup of $\mathbb{Z}_{5p}$ generated by t, then X has a Hamilton cycle. Moreover, if T contains one element of order p and another element of order 5, then their product generates $\mathbb{Z}_{5p}$, and so X has a Hamilton cycle. We can therefore assume that $\langle T \setminus 0\rangle$ is either empty or it is one of $\langle 5\rangle$ and $\langle p\rangle$.

As $X\langle B\rangle$ is an independent set for each $B \in \mathcal{B}$, there is no element of order 5 in S or in R. If $\langle S\rangle = \mathbb{Z}_{5p}$ and $\langle R\rangle = \mathbb{Z}_{5p}$, then the subgraphs induced on each of the orbits of φ are connected vertex-transitive graphs of order $5p$, and so they both contain a Hamilton cycle. Clearly, X has a Hamilton path in this case. With no loss of generality we can thus assume that $\langle S\rangle \neq \mathbb{Z}_{5p}$. This implies that $S = \emptyset$ or $\langle S\rangle = \mathbb{Z}_5$. Suppose first that $S = \emptyset$. Then regularity of X implies that $R = \emptyset$ as well. But then, by the above remarks on T, X is not connected, a contradiction. Therefore, $\langle S\rangle = \mathbb{Z}_5$. As X is regular, we have that $|S| = |R|$, and so either $\langle R\rangle = \mathbb{Z}_5$ or $\langle R\rangle = \mathbb{Z}_{5p}$. In the former case the subgraph induced on each of the orbits of ρ contains a p-cycle. Moreover, the facts that $T \ncong \mathbb{Z}_{5p}$ and that X is connected combined together imply that there exists some $t \in T$ of order 5, and so t and 0 give rise to a 10-cycle of X_ρ. Therefore, X has a Hamilton path in this case. We are left with the possibility $\langle R \rangle = \mathbb{Z}_{5p}$. In view of the fact that no element of order 5 exists in R, some $r \in R$ such that $\langle r \rangle = \mathbb{Z}_{5p}$ exists. We can assume that $r = 1$ (otherwise take φ^r instead of φ). Since $\langle S \rangle = \langle 5 \rangle$, we have $5k \in S$ for some $k \in \{1, 2, \ldots, p-1\}$. Thus X contains a subgraph isomorphic to the generalized Petersen graph $GP(5p, 5k)$ which has a Hamilton cycle (see [2]).

CASE 3. $(p : 2 : 5)$.

Then G/K acts on $X_{\mathcal{B}}$ imprimitively with p blocks of size 2, and by Proposition 2.1, there exists a transitive subgroup H/K of G/K with blocks of size p. Therefore there exists a transitive subgroup of $\mathrm{Aut}(X)$ such that with respect to this subgroup X is of type $(2 : p : 5)$, and so, by Case 2, X has a Hamilton path. $\square$

Lemma 3.5. *If the size of blocks in $\mathcal{B}$ is 10, then X has a Hamilton path.*

Proof. Note that $X_{\mathcal{B}}$ is a connected p-circulant, and so, by Proposition 2.6, $X_{\mathcal{B}}$ is edge-Hamiltonian. Proposition 2.15 implies that N^B is a simple group of degree 10 for every $B \in \mathcal{B}$. By [45], the only transitive simple groups of degree 10 up to permutation isomorphism are the alternating groups A_{10}, A_6 and A_5. Since in each of these three groups subgroups of index 10 are maximal we can conclude that all these groups are primitive, and thus N^B is a primitive group of degree 10.

Suppose first that there exist two adjacent blocks $B, B' \in \mathcal{B}$ such that $X[B, B']$ is of valency no less than 3. Let C be a p-cycle in $X_{\mathcal{B}}$ that contains the edge BB' (such a cycle exists since $X_{\mathcal{B}}$ is edge-Hamiltonian). Since the valency of $X[B, B']$ is no less than 3, there exist at least two edges with non-zero voltage, denote them by i and j, $i, j \in \mathbb{Z}_{10}$. If $(i, 10) = 1$, then the lift of C is clearly a cycle of length $10p$, and thus a Hamilton cycle of X. If $(i, 10) = 2$, then C lifts to two $5p$-cycles, and the connectivity of X implies that X has a Hamilton path. If, however, $(i, 10) = 5$, then $(j, 10) \neq 5$. Namely, if also $(j, 10) = 5$, then X has multiedges, which is not possible since X is a simple graph. Thus, either $(j, 10) = 1$ or $(j, 10) = 2$. In both cases X clearly contains a Hamilton path.

We may now assume that the valency between any two adjacent blocks is less than 3. If there exist two adjacent blocks $B, B' \in \mathcal{B}$, such that $X[B, B']$ has valency 2. Since $X[B, B']$ is vertex-transitive, it follows that $X[B, B']$ is isomorphic to one of the following graphs: C_{20}, $2C_{10}$ and $5C_4$. However, since N^B is primitive only the first case can occur, in particular, $X[B, B'] \cong C_{20}$. Since $X[B, B']$ is of valency 2, there must be one edge with non-zero voltage, denote this voltage by $i \in \mathbb{Z}_{10}$. Since $X[B, B'] = C_{20}$ we have that $(i, 10) = 1$. Since $X_{\mathcal{B}}$ is edge-Hamiltonian, there exists a Hamilton cycle in $X_{\mathcal{B}}$ containing the edge BB', and thus one can easily see that X has a Hamilton cycle in this case.

We may therefore assume that for any two adjacent blocks $B, B' \in \mathcal{B}$ the bipartite graph $X[B, B']$ is of valency 1, in particular, $X[B, B'] \cong 10K_2$. If $X\langle B \rangle$ is a connected graph, then we can easily see that there is a Hamilton path in X. If $X\langle B \rangle$ is disconnected, then $X\langle B \rangle \in \{2C_5, 2K_5, 5K_2, 10K_1\}$. However, since N^B is primitive we must have $X\langle B \rangle \cong 10K_1$. Since N^B is isomorphic to A_5, A_6 or A_{10} there exists a nontrivial automorphism $\alpha \in N$ such that α fixes a vertex in B. But then, since $X[B, B'] \cong 10K_2$ and $\mathcal{B}$ is an imprimitivity block system of G arising from orbits of a normal subgroup N of G, the connectivity of X implies that α fixes all the vertices of X, a contradiction. $\square$

Lemma 3.6. *If $p > 7$ and the size of blocks in $\mathcal{B}$ is $2p$, then X has a Hamilton path.*

Proof. Note that either $X_{\mathcal{B}} \cong K_5$ or $X_{\mathcal{B}} \cong C_5$. Let $\mathcal{B} = \{B_i \mid i \in \mathbb{Z}_5\}$. Since $p > 7$, by Proposition 2.2, there exists a $(10, p)$-semiregular automorphism $\rho \in G$. Let $\mathcal{S} = \{S_i, S'_i \mid i \in \mathbb{Z}_5\}$ be the set of its orbits. By Proposition 2.3, each block in $\mathcal{B}$ is a union of two orbits of ρ. With no loss of generality we can assume that $B_0 = S_0 \cup S'_0$, $B_1 = S_1 \cup S'_1$, $B_2 = S_2 \cup S'_2$, $B_3 = S_3 \cup S'_3$ and $B_4 = S_4 \cup S'_4$.

Consider the subgraph $\bar{X}_{\mathcal{S}}$ of $X_{\mathcal{S}}$, which is obtained from $X_{\mathcal{S}}$ by deleting the edges $S_iS'_i$, $i \in \mathbb{Z}_5$ (if they exist). Observe that for any two adjacent blocks $B, B' \in \mathcal{B}$ we have that either $X_{\mathcal{S}}[B, B'] \cong K_4$ or $X_{\mathcal{S}}[B, B'] \cong 2K_2$.

Suppose that there exist $B, B' \in \mathcal{B}$ such that $X_{\mathcal{S}}[B, B'] \cong K_4$. Suppose that there also exists a pair of adjacent blocks $D, D' \in \mathcal{B}$ such that $X_{\mathcal{S}}[D, D'] \cong 2K_2$. Then, since for any two edges in $X_{\mathcal{B}} \in \{K_5, C_5\}$ there exists a Hamilton cycle of $X_{\mathcal{B}}$ containing both of these two edges, there exists a Hamilton cycle of $X_{\mathcal{B}}$ containing both edges BB' and DD', and thus this cycle gives rise to a Hamilton cycle of $X_{\mathcal{S}}$. Moreover, in view of regularity of X and regularity of the subgraphs $X\langle B\rangle$, $B \in \mathcal{B}$, this cycle contains a multiedge and so, by Proposition 2.10, X has a Hamilton cycle. We may therefore assume that the bipartite graphs $X_{\mathcal{S}}[B, B']$, $B, B' \in \mathcal{B}$, are pairwise isomorphic, in particular, either for any two adjacent blocks $B, B' \in \mathcal{B}$ we have $X_{\mathcal{S}}[B, B'] \cong K_4$ or for any two adjacent blocks $B, B' \in \mathcal{B}$ we have $X_{\mathcal{S}}[B, B'] \cong 2K_2$.

Below it will be convenient to have the following notation. For two adjacent blocks $B_i, B_j \in \mathcal{B}$ we will say that the bipartite subgraph $X_{\mathcal{S}}[B_i, B_j]$ is of *type* 0, of *type* 1 and of *type* 2 if, respectively, $E(X_{\mathcal{S}}[B_i, B_j]) = \{S_iS_j, S_i'S_j'\}E(X_{\mathcal{S}}[B_i, B_j]) = \{S_iS_j', S_i'S_j\}$ and $E(X_{\mathcal{S}}[B_i, B_j]) = \{S_iS_j, S_iS_j', S_i'S_j, S_i'S_j'\}$. We will say that an edge in $X_{\mathcal{B}}$ is of type k if the corresponding bipartite subgraph in $X_{\mathcal{S}}$ is of type k. Note that, by the above paragraph, either all edges of $X_{\mathcal{B}}$ are of type 2 or they are all of type different from type 2. Moreover, any 5-cycle $C = u_0u_1u_2u_3u_4u_0$, $\{u_i \mid i \in \{0, 1, 2, 3, 4\}\} \subseteq V(X_{\mathcal{B}})$, in $X_{\mathcal{B}}$ can be represented by a vector $[i_0, i_1, i_2, i_3, i_4]$, where i_j is the type of the edge u_ju_{j+1}. In addition, we will say that the cycle C is of *type* $[i_0, i_1, i_2, i_3, i_4]$.

Now suppose that $X_{\mathcal{B}}$ does not contain edges of type 2. Also, suppose that $X_{\mathcal{B}} \cong K_5$. Then the edge set of $X_{\mathcal{B}}$ can be viewed as the set of two disjoint 5-cycles, say $\mathcal{C}$ and $\mathcal{C}'$. Without loss of generality we may assume that one of these 5-cycles, say $\mathcal{C}$, is of type $[\tau, 0, 0, 0, 0]$. If $\tau = 0$, then (for symmetry reasons) we may assume that $\mathcal{C}'$ is of one of the following types: $[1, 0, 0, 0, 0]$, $[1, 1, 0, 0, 0]$, $[1, 0, 1, 0, 0]$, $[1, 1, 1, 0, 0]$, $[1, 1, 0, 1, 0]$, $[1, 1, 1, 1, 0]$, $[1, 1, 1, 1, 1]$, or $[0, 0, 0, 0, 0]$. These give eight possibilities for the graph $\bar{X}_{\mathcal{S}}$. If, however, $\tau = 1$, then by a detailed consideration of all possible types for $\mathcal{C}'$, one can see that we get eight more possibilities for the graph $\bar{X}_{\mathcal{S}}$. In Fig. 3 we show all these possibilities in the graph $X_{\mathcal{B}}$, whereas in Fig. 4 we show all possible graphs $\bar{X}_{\mathcal{S}}$. In particular, for $X_{\mathcal{B}} \cong K_5$, $\bar{X}_{\mathcal{S}}$ is isomorphic to one of the graphs Y_i, $i \in \{0, 1, 2, \ldots, 15\}$. In addition, if $X_{\mathcal{B}} \cong C_5$, then we can clearly assume that $\bar{X}_{\mathcal{S}}$ is isomorphic to one of the graphs Y_{16} and Y_{17} in Fig. 4 (see also Fig. 3). If, however, $X_{\mathcal{B}}$ contains an edge of type 2, then all edges in $X_{\mathcal{B}}$ are of this type, and thus only two more possibilities occur. Let us denote the graph arising from this case by Y_{18} if $X_{\mathcal{B}} \cong K_5$ and by Y_{19} if $X_{\mathcal{B}} \cong C_5$.

Observe that $Y_0 \cong Y_{13} \cong Y_{14}$, $Y_1 \cong Y_{11} \cong Y_{15}$, $Y_2 \cong Y_5 \cong Y_9 \cong Y_{10}$, and $Y_3 \cong Y_6$. We may therefore assume that $\bar{X}_{\mathcal{S}}$ is isomorphic to one of the following twelve graphs: $Y_0, Y_1, Y_2, Y_3, Y_4, Y_7, Y_8, Y_{12}, Y_{16}, Y_{17}, Y_{18}$ or Y_{19}.

If $X\langle B_0\rangle$ is a connected graph, then for each of its vertices there exists a Hamilton path of $X\langle B_0\rangle$ starting at that vertex, so X clearly has a Hamilton path in this case. We can thus assume that $X\langle B_0\rangle$ is not connected. As it is a vertex-transitive graph, it is isomorphic to $2pK_1$, to pK_2 or it is a disjoint union of two isomorphic connected p-circulants. We consider each of the three cases separately.

Case 1. $X\langle B_0\rangle \cong 2pK_1$.

Since X is connected, the quotient graph $X_{\mathcal{S}} = \bar{X}_{\mathcal{S}}$ is isomorphic to one of the graphs Y_i, $i \in \{0, 1, 2, 3, 4, 8, 12, 17, 18, 19\}$. Then any edge of $X_{\mathcal{S}}$ lies on some Hamilton cycle of $X_{\mathcal{S}}$ and thus Proposition 2.10 implies that we can assume that no multiedge exists in X_ρ. By considering all Hamilton cycles in $X_{\mathcal{S}}$ one can easily see that the connectedness of X forces some Hamilton cycle of $X_{\mathcal{S}}$, whose lift contains a Hamilton cycle of X, to exist. The details are left to the reader.

Case 2. $X\langle B_0\rangle \cong pK_2$.

It is clear that $X[S_0, S_1] \cong pK_2$. Suppose first that

$$\bar{X}_{\mathcal{S}} \cong Y_i, \quad \text{where } i \in \{0, 1, 2, 3, 4, 8, 12, 17, 18, 19\}.$$

Then, by Case 1, we may assume that no multiedge exists in $\bar{X}_\rho$, and moreover that all the edges in $\bar{X}_{\mathcal{S}}$ carry label 0. Observe also that in all cases there exists a 10-cycle C in $\bar{X}_{\mathcal{S}}$ such that the endvertices of the edges S_iS_i', $i \in \mathbb{Z}_5$, are antipodal vertices on the cycle C in $X_{\mathcal{S}}$. Note also that in all cases for any edge S_iS_i', $i \in \mathbb{Z}_5$, there exists a Hamilton cycle of $X_{\mathcal{S}}$ containing this edge, and therefore, by Proposition 2.10, we may assume that there is no multiedge in X_ρ. Also, if there exists a Hamilton cycle C of $X_{\mathcal{S}}$ such that $l(C) \neq 0$, then X has a Hamilton cycle. Therefore, we can assume that no such Hamilton cycle of $X_{\mathcal{S}}$ exists.

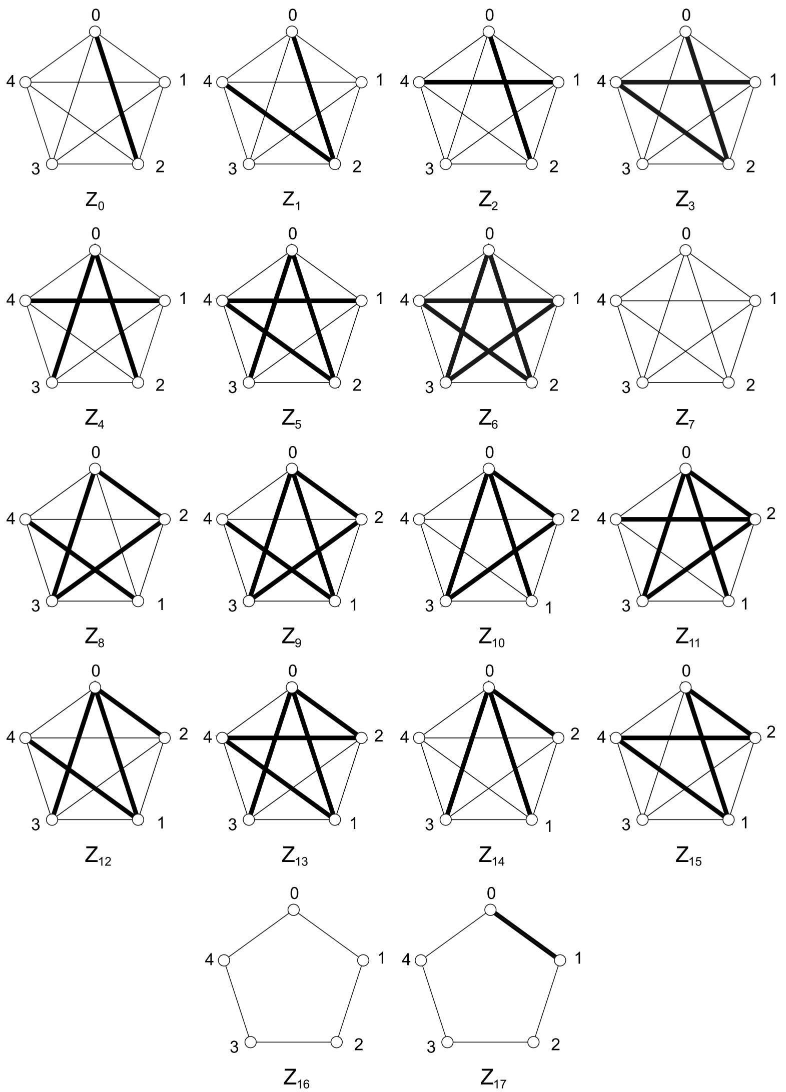

Fig. 3. All possible structures of $\bar{X}_{\mathcal{S}}$ shown in $X_{\mathcal{B}}$ if $\bar{X}_{\mathcal{S}}$ does not contain edges of type 2. Bold edges are edges of type 1.

Let us relabel the vertices of $X_{\mathcal{S}}$ in such a way that $C = u_0u_1u_2u_3u_4u_5u_6u_7u_8u_9u_0$ and let the label of the arc u_iu_{i+5}, $i \in \mathbb{Z}_5$, be denoted by a_i. Since

$$u_0u_5u_4u_3u_2u_1u_6u_7u_8u_9u_0$$
$$u_0u_1u_6u_5u_4u_3u_2u_7u_8u_9u_0$$
$$u_0u_1u_2u_7u_6u_5u_4u_3u_8u_9u_0$$
$$u_0u_1u_2u_3u_8u_7u_6u_5u_4u_9u_0$$
$$u_0u_1u_2u_3u_4u_9u_8u_7u_6u_5u_0$$

are Hamilton cycles in $X_{\mathcal{S}}$ with labels $a_0 + a_1$, $a_1 + a_2$, $a_2 + a_3$, $a_3 + a_4$, $a_4 + a_0$, respectively, we have that $a_0 = a_1 = a_2 = a_3 = a_4 \neq 0$. However, $u_0u_5u_6u_1u_2u_7u_8u_3u_4u_9u_0$ is a Hamilton cycle of $X_{\mathcal{S}}$ whose label is equal to a_0, and so $a_0 = 0$, a contradiction.

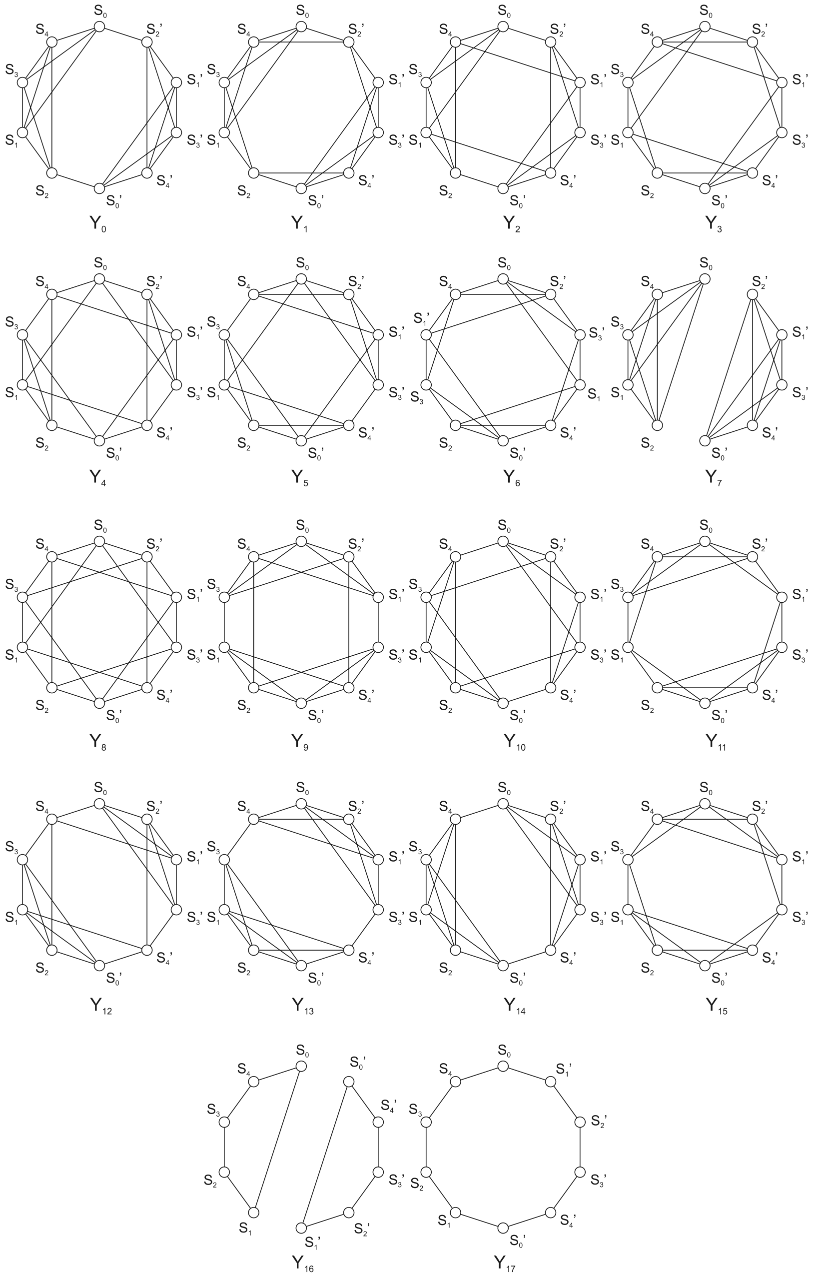

Fig. 4. All possible graphs for $\bar{X}_{\mathcal{S}}$ where the graph Y_i corresponds to the graph Z_i in Fig. 3.

Suppose now that $\bar{X}_{\mathcal{S}} \cong Y_i$, $i \in \{7, 16\}$. Then every edge of $X_{\mathcal{S}}$ is contained on some Hamilton cycle of $X_{\mathcal{S}}$, and so Proposition 2.10 implies that we can assume that no multiedge exists in X_ρ. By considering all Hamilton cycles in $X_{\mathcal{S}}$ one can easily see that the connectedness of X forces some Hamilton cycle of $X_{\mathcal{S}}$, whose lift contains a Hamilton cycle of X, to exist. The details are left to the reader.

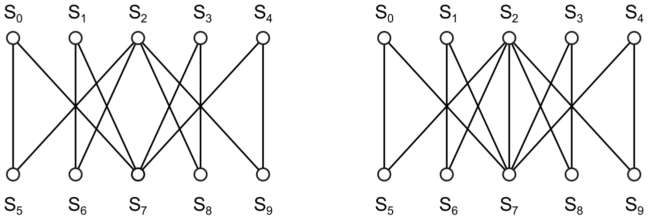

Fig. 5. The two bipartite graphs of order 10 and minimal valency 2 not possessing a disjoint union of two cycles such that the union of their vertices covers the vertex set of the graph.

Case 3. $X\langle B_0\rangle$ is isomorphic to a disjoint union of two isomorphic connected p-circulants.

In view of connectedness of X the quotient graph $X_{\mathcal{S}} = \bar{X}_{\mathcal{S}}$ is isomorphic to

$$Y_i, \quad i \in \{0, 1, 2, 3, 4, 8, 12, 17, 18, 19\}.$$

As the p-circulants are precisely the graphs $X\langle S_i\rangle$, where $i \in \mathbb{Z}_{10}$, a Hamilton path exists in X. This completes the proof. □

Lemma 3.7. *If $p > 7$ and the size of blocks in $\mathcal{B}$ is $5p$, then X has a Hamilton path.*

Proof. Note that $|\mathcal{B}| = 2$ and $X_{\mathcal{B}} \cong K_2$. Let us denote the two blocks of $\mathcal{B}$ by B and B'. By Proposition 2.2 there exists a $(10, p)$-semiregular automorphism $\rho \in G$ of X. Let $\mathcal{S} = \{S_i \mid i \in \mathbb{Z}_{10}\}$ be the set of its orbits. By Proposition 2.3 each block in $\mathcal{B}$ is a union of five orbits of ρ. With no loss of generality we can assume that $B = S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4$ and $B' = S_5 \cup S_6 \cup S_7 \cup S_8 \cup S_9$.

By Proposition 2.15, for every $B \in \mathcal{B}$ the group N^B is simple. In addition, by Proposition 2.19, either N^B is primitive, or $N^B \in \{\mathrm{PSL}(2, 11), \mathrm{PSL}(m, q)\}$, where m is a prime and q is a prime power. The lemma will follow from the five claims given below. Throughout the proof we will frequently use the following fact about the number of edges between orbits of ρ in the subgraph $\bar{X}_{\mathcal{S}}$ of $X_{\mathcal{S}}$, which is obtained from $X_{\mathcal{S}}$ by deleting the edges between the orbits inside the blocks B and B' (if they exist):

$$\sum_{j\in\{5,\ldots,9\}} d(S_i, S_j) = \sum_{j\in\{0,\ldots,4\}} d(S_j, S_k) \quad \text{for every } i \in \{0, 1, 2, 3, 4\} \text{ and } k \in \{5, 6, 7, 8, 9\}. \tag{1}$$

Claim 1. *If $X\langle B\rangle \cong X\langle B'\rangle$ is connected, then X contains a Hamilton path.*

Since $X\langle B\rangle \cong X\langle B'\rangle$ is a connected vertex-transitive graph of order $5p$, by Proposition 2.8, it has a Hamilton cycle, and thus, since X is connected, we can conclude that X contains a Hamilton path.

Claim 2. *If $X\langle B\rangle \cong X\langle B'\rangle = 5pK_1$, then X contains a Hamilton path.*

Since $X\langle B\rangle \cong X\langle B'\rangle \cong 5pK_1$ the graph $\bar{X}_{\mathcal{S}} = X_{\mathcal{S}}$ is a connected bipartite graph of order 10 with bipartition sets of size 5. Moreover, (1) implies that its minimal valency is not less than 2. From the list of all bipartite graphs of order 10, given in [40], we get, with the help of the program package Magma [5], that there exist 600 (of which five are regular) nonisomorphic bipartite graphs of order 10 with bipartition sets of size 5 and minimal valency no less than 2. We consider two cases depending on whether $X_{\mathcal{S}}$ is irregular or regular.

Case 2.1, X_S is irregular.

Subcase 2.1.1. $X_{\mathcal{S}}$ possesses a Hamilton cycle.

In every such graph one can, with the help of Magma, find such a Hamilton cycle that in the corresponding multigraph this Hamilton cycle contains a multiedge, and thus, by Proposition 2.10, X contains a Hamilton cycle.

Subcase 2.1.2. $X_{\mathcal{S}}$ does not possess a Hamilton cycle.

With the exception of the two graphs shown in Fig. 5 every graph belonging to this subfamily contains a disjoint union of a 4-cycle and a 6-cycle such that in the corresponding multigraph these

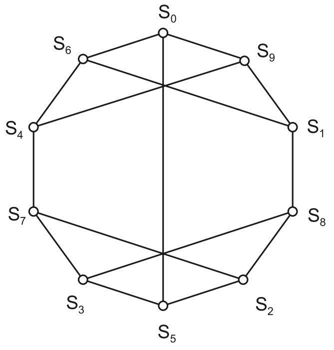

Fig. 6. The graph $K_{5,5} - (C_6 \cup C_4)$.

two cycles both contain a multiedge. Therefore these two cycles lift to a disjoint union of a $4p$-cycle and $6p$-cycle in X, implying that X contains a Hamilton path.

We may now assume that $X_{\mathcal{S}}$ is one of the two graphs shown in Fig. 5. We will show that both cases lead to a contradiction.

Let the vertices of X be labeled in such a way that $S_i = \{s_i^j \mid j \in \mathbb{Z}_p\}$, $i \in \mathbb{Z}_{10}$. Suppose first that $X_{\mathcal{S}}$ is the graph shown in the left-hand side picture of Fig. 5. Then $X_{\mathcal{S}}$ has eight vertices of valency 2 and two vertices of valency 4. Since X is regular the edges S_iS_{i+5}, $i \in \mathbb{Z}_{10} \setminus \{2, 7\}$, are multiedges in X_ρ. Hence the 6-cycles

$$S_2S_5S_0S_7S_1S_6S_2 \quad \text{and} \quad S_2S_9S_4S_7S_3S_8S_2$$

in X_ρ both lift to a $6p$-cycle in X. Consequently, each of the vertices s_i^j, where $i \in \mathbb{Z}_{10} \setminus \{2, 7\}$ and $j \in \mathbb{Z}_p$, is contained on at least one $6p$-cycle. On the other hand, since the above 6-cycles in X_ρ both contain S_2 and S_7, it follows that the vertices s_2^j and s_7^j, $j \in \mathbb{Z}_p$, are contained on at least two different $6p$-cycles in X. Now vertex-transitivity of X implies that also the vertices s_i^j, where $i \in \mathbb{Z}_{10} \setminus \{2, 7\}$ and $j \in \mathbb{Z}_p$, are contained on at least two different $6p$-cycles. But since any $6p$-cycle in X containing a vertex s_i^j, $i \in \mathbb{Z}_{10} \setminus \{2, 7\}$ and $j \in \mathbb{Z}_p$, and not arising from the above mentioned 6-cycles in X_ρ, must contain at least one vertex from S_7 (respectively, S_2), vertices from the orbits S_i, $i \in \mathbb{Z}_{10} \setminus \{2, 7\}$ lie on less $6p$-cycles than those from the orbits S_2 and S_7. But this is clearly impossible in view of vertex-transitivity of X. That the other case (when $X_{\mathcal{S}}$ is isomorphic to the graph shown in the right-hand side picture of Fig. 5) is not possible can be proved with a similar argument. The details are left to the reader.

Case 2.2. X_S is a regular graph.

There are five regular bipartite graphs of order 10 with the two bipartition sets of size 5: C_{10}, $K_{5,5}$, $K_{5,5} - 5K_2$, C_{10}^+, and $K_{5,5} - (C_6 \cup C_4)$. Observe that the first four of these graphs are vertex-transitive graphs, and thus the same argument as in the proof of Lemma 3.3 applies. We may, therefore, assume that $X_{\mathcal{S}} = K_{5,5} - (C_6 \cup C_4)$. Let the vertices of $X_{\mathcal{S}}$ be labeled in such a way as shown in Fig. 6. Since

$$S_0S_9S_1S_8S_2S_5S_3S_7S_4S_6S_0$$

is a Hamilton cycle of $X_{\mathcal{S}}$, by Proposition 2.10, we can assume that all the edges on this cycle are single edges in X_ρ. Moreover, we can assume that all the edges on this cycle carry label 0. Further, since every edge of $X_{\mathcal{S}}$ lies on some Hamilton cycle of $X_{\mathcal{S}}$ we can assume that $X_{\mathcal{S}} = X_\rho$, that is, no multiedge exists in X_ρ. Next, since

$$S_0S_5S_3S_8S_2S_7S_4S_6S_1S_9S_0,$$
$$S_0S_5S_2S_7S_3S_8S_1S_6S_4S_9S_0,$$
$$S_0S_5S_3S_7S_2S_8S_1S_6S_4S_9S_0,$$

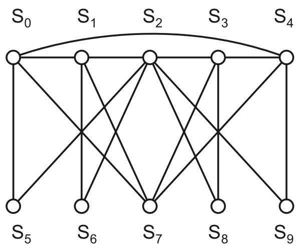

Fig. 7. The graph $\tilde{X}_{\mathcal{S}}$ in case $\bar{X}_{\mathcal{S}}$ is isomorphic to the graph shown on the left-hand side picture of Fig. 5.

$$S_0S_6S_1S_8S_2S_5S_3S_7S_4S_9S_0,$$
$$S_0S_6S_4S_7S_2S_5S_3S_8S_1S_9S_0,$$

are Hamilton cycles in $X_{\mathcal{S}}$, by Proposition 2.10, we can assume that for the labels of the arcs of $X_{\mathcal{S}}$ the following equations hold:

$$l(S_0S_5) + l(S_3S_8) + l(S_2S_7) + l(S_6S_1) = 0,$$
$$l(S_0S_5) + l(S_2S_7) + l(S_3S_8) + l(S_1S_6) + l(S_4S_9) = 0,$$
$$l(S_0S_5) + l(S_7S_2) + l(S_1S_6) + l(S_4S_9) = 0,$$
$$l(S_6S_1) + l(S_4S_9) = 0,$$
$$l(S_7S_2) + l(S_3S_8) = 0.$$

Combining together these equations one can easily get that $l(S_0S_5) = l(S_1S_6) = l(S_2S_7) = l(S_3S_8) = l(S_4S_9) = 0$ (using the fact that for $k \in \mathbb{Z}_p$, $p > 7$ a prime, we have $3k \equiv 0 \pmod p$ if and only if $k = 0$), and thus X is disconnected, a contradiction.

Claim 3. *If $X\langle B\rangle \cong X\langle B'\rangle \cong pC_5$, then X contains a Hamilton path.*

Observe that 5-cycles in the blocks B and B' form an imprimitivity block system $\mathcal{C}$ of G. Hence Proposition 2.3 implies that either $S_i \cap C = \emptyset$ or $|S_i \cap C| = 1$ for every $i \in \mathbb{Z}_p$ and every $C \in \mathcal{C}$. Since $p > 7$ it follows that $X\langle B\rangle_{\mathcal{S}} \cong X\langle B'\rangle_{\mathcal{S}} \cong C_5$.

The graph $\bar{X}_{\mathcal{S}}$ obtained from $X_{\mathcal{S}}$ by deleting the edges inside the blocks B and B' is clearly a bipartite graph of order 10 with each bipartition set of size 5. (Note that $\bar{X}_{\mathcal{S}}$ can be disconnected.) Checking the list of all bipartite graphs of order 10 given in [40], and using (1), one can see that either $\bar{X}_{\mathcal{S}}$ is isomorphic to the graph shown in the left-hand side picture of Fig. 5 or $\bar{X}_{\mathcal{S}}$ contains $5K_2$.

Case 3.1. $\bar{X}_{\mathcal{S}}$ is isomorphic to the graph shown in the left-hand side picture of Fig. 5.

Then we can, without loss of generality, assume that the graph $\tilde{X}_{\mathcal{S}}$ obtained from $X_{\mathcal{S}}$ by deleting the edges in B' is the graph shown in Fig. 7. Also, the regularity of X and $X\langle B\rangle \cong X\langle B'\rangle$ combined together imply that $d(S_0, S_5) > 1$, and consequently any Hamilton cycle of $X_{\mathcal{S}}$ containing this edge, by Proposition 2.10, gives rise to a Hamilton cycle of X.

Since $X\langle B'\rangle \cong C_5$, the vertex S_7 is adjacent to two of the vertices from $\{S_5, S_6, S_8, S_9\}$. In particular, we can assume (for symmetry reasons) that one of the following occurs in $X_{\mathcal{S}}$:

(i) $S_7S_8, S_7S_6 \in E(X_{\mathcal{S}})$;
(ii) $S_7S_8, S_7S_9 \in E(X_{\mathcal{S}})$;
(iii) $S_7S_8, S_7S_5 \in E(X_{\mathcal{S}})$;
(iv) $S_7S_9, S_7S_5 \in E(X_{\mathcal{S}})$.

If (i) occurs, then $S_0S_5S_2S_9S_4S_3S_8S_7S_6S_1S_0$ is a Hamilton cycle of $X_{\mathcal{S}}$ containing a multiedge in X_ρ. Next, if (ii) occurs, then since $X\langle B'\rangle_{\mathcal{S}} \cong C_5$, we have that S_8 is either adjacent to S_5 or it is adjacent to S_6. For the first case, $S_0S_5S_8S_7S_9S_4S_3S_2S_6S_1S_0$ is a Hamilton cycle of $X_{\mathcal{S}}$ containing a multiedge in X_ρ. For the latter case, $S_0S_1S_6S_8S_7S_9S_4S_3S_2S_5S_0$ is a Hamilton cycle of $X_{\mathcal{S}}$ containing a multiedge in X_ρ. Further, if (iii) occurs, then $S_4S_9S_2S_6S_1S_0S_5S_7S_8S_3S_4$ is a Hamilton cycle of $X_{\mathcal{S}}$ containing a multiedge in X_ρ.

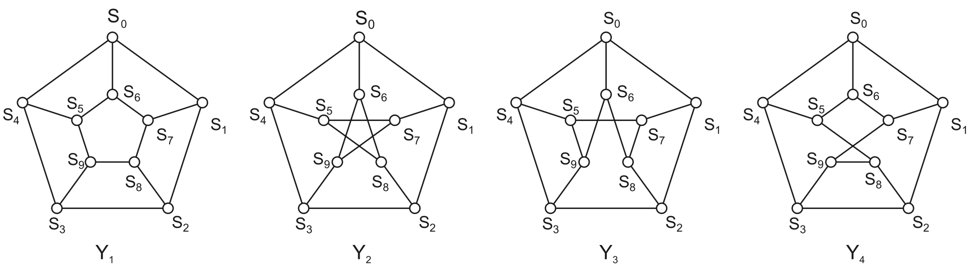

Fig. 8. Possibilities for a subgraph of $X_{\mathcal{S}}$ in case $\bar{X}_{\mathcal{S}}$ contains $5K_2$.

Finally, if (iv) occurs, then $S_1S_6S_2S_8S_3S_4S_9S_7S_5S_0S_1$ is a Hamilton cycle of $X_{\mathcal{S}}$ containing a multiedge in X_ρ. Therefore in all these cases Proposition 2.10 applies.

Case 3.2. $\bar{X}_{\mathcal{S}}$ contains $5K_2$.

Then $X_{\mathcal{S}}$ contains one of the four graphs Y_i, $i \in \{1, 2, 3, 4\}$ shown in Fig. 8, and thus four cases need to be considered. However, recall that all the edges in $X\langle B\rangle_{\mathcal{S}}$ and $X\langle B'\rangle_{\mathcal{S}}$ are single edges in X_ρ, and moreover, each edge in these two subgraphs carries label 0.

Subcase 3.2.1. $X_{\mathcal{S}}$ contains Y_1.

Observe that every edge of Y_1 is contained in a Hamilton cycle and thus we may assume that all the edges of Y_1 are single edges in X_ρ. Further, since

$$C = S_0S_6S_5S_9S_8S_7S_1S_2S_3S_4S_0$$

is a Hamilton cycle of Y_1, we can assume that it carries label 0. Now, observe that with permuting the indices of the orbits S_i in C with the permutation $(0\ 1\ 2\ 3\ 4)(6\ 7\ 8\ 9\ 5)$ we get four more Hamilton cycles in Y_1. Consequently, we can assume that all the edges of Y_1 carry label 0. Since X is connected, it follows that Y_1 is a proper subgraph of X_S, implying that there must exist an arc e in X_S carrying a non-zero label, and without loss of generality we can assume that S_0 is an endvertex of this arc. Since edges of Y_1 are single edges in X_ρ and Y_1 is symmetric we can assume that either $e = S_0S_5$ or $e = S_0S_8$. In the first case $S_0S_5S_6S_7S_1S_2S_8S_9S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label, and in the second case $S_0S_8S_9S_5S_6S_7S_1S_2S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label. Thus we can conclude that in both cases X has a Hamilton cycle.

Subcase 3.2.2. $X_{\mathcal{S}}$ contains Y_2 (the Petersen graph).

Recall that all the edges in $X\langle B\rangle_{\mathcal{S}}$ and $X\langle B'\rangle_{\mathcal{S}}$ are single edges in X_ρ, and moreover, each edge in these two subgraphs carries label 0. Since for any pair of edges from the set $A = \{S_0S_6, S_1S_7, S_2S_8, S_3S_9, S_4S_5\}$ there exists a disjoint union of two 5-cycles we can assume that at most one edge from A is a multiedge in X_ρ. (Namely, if two such edges exist in Y_2, then we have two disjoint 5-cycles in Y_2, each containing one of these edges, and thus they both give rise to a $5p$-cycle in X, implying that X has a Hamilton path.) If, however, exactly one of the edges from A is a multiedge in X_ρ, say that this edge is the edge S_0S_6, then the regularity of X implies that S_1 is an endvertex of an edge of $X_{\mathcal{S}}$ which is not contained in Y_2 and is not incident to either of the vertices S_0 and S_6. This shows that S_1S_8 or S_1S_9 or S_1S_5 is an edge of $X_{\mathcal{S}}$. In each of these cases one can find a Hamilton cycle of X_ρ containing the multiedge S_0S_6, and thus Proposition 2.10 implies that X contains a Hamilton cycle. In particular,

(a) if $S_1S_8 \in E(X_{\mathcal{S}})$, then $S_0S_6S_8S_1S_2S_3S_9S_7S_5S_4S_0$ is a Hamilton cycle of X_ρ containing the multiedge S_0S_6;
(b) if $S_1S_9 \in E(X_{\mathcal{S}})$, then $S_0S_6S_8S_5S_7S_9S_1S_2S_3S_4S_0$ is a Hamilton cycle of X_ρ containing the multiedge S_0S_6;
(c) if $S_1S_5 \in E(X_{\mathcal{S}})$, then $S_0S_6S_8S_2S_1S_5S_7S_9S_3S_4S_0$ is a Hamilton cycle of X_ρ containing the multiedge S_0S_6.

We may therefore assume that no edge in A is a multiedge in X_ρ. Let the labels of the arcs $S_0S_6, S_1S_7, S_2S_8, S_3S_9, S_4S_5$ be denoted, respectively, by a, b, c, d and e. Observe that if there exist two disjoint 5-cycles in Y_2, whose lifts both contain a $5p$-cycle, then the connectedness of X implies that

X has a Hamilton path. We can thus assume that no two such 5-cycles exist in $X_{\mathcal{B}}$. Considering all possible disjoint 5-cycles in Y_2 we have

$$a = c \quad \text{or} \quad d = e, \qquad b = d \quad \text{or} \quad a = e, \qquad c = e \quad \text{or} \quad a = b,$$
$$a = d \quad \text{or} \quad b = c, \qquad b = e \quad \text{or} \quad c = d.$$

Assume first that we have $a = b = c = d = e$. Then, since X is connected we get that Y_2 is a proper subgraph of $X_{\mathcal{S}}$. In particular, there exists a vertex which is an endvertex of an edge of $X_{\mathcal{S}}$ such that it is not contained in Y_2 and that the arc with the tail in this vertex carries a label $t \notin \{0, a\}$. Since Y_2 is symmetric we can assume that such a vertex is the vertex S_1, and thus for the edge having S_1 for one of its endvertices we again have possibilities (a)–(c) listed above. However, in this case also S_1S_6 can be such an edge. But since all the edges in Y_2 are single edges in X_ρ, we have (in view of the symmetry of Y_2) that it suffices to consider possibilities (a)–(c). But in all these cases one can easily see that since $t \notin \{0, a\}$ the listed Hamilton cycles all give rise to a Hamilton cycle of X also in this case.

Assume now that not all labels a, b, c, d and e are equal. With no loss of generality assume that $a \neq b$, and so $c = e$. Suppose first that $a = d$. Then $b \neq d$, and so $d = a = e = c$. The reader may check that then the vertices of S_1 are contained on precisely two 5-cycles arising from Y_2, whereas the vertices of S_0 are contained on precisely four 5-cycles arising from Y_2, which in view of vertex-transitivity of X implies that Y_2 is a proper subgraph of $X_{\mathcal{S}}$. In particular, since edges in Y_2 are single edges it follows that each vertex of $X_{\mathcal{S}}$ lies on an edge that is not contained in Y_2. Consider all possibilities for such an edge with endvertex S_1. Let $t \in \mathbb{Z}_p$ be the label of the corresponding arc with the tail in S_1. For symmetry reasons (since $d = a = e = c$) it suffices to assume that either $S_1S_8 \in E(X_{\mathcal{S}})$ or $S_1S_9 \in E(X_{\mathcal{S}})$.

First, suppose that $S_1S_8 \in E(X_{\mathcal{S}})$. Then whenever $t \neq a$ and $a \neq 0$ the Hamilton cycle given in (a) lifts to a Hamilton cycle of X. Thus we may assume that $t = a$ (in addition, S_1S_8 is not a multiedge in X_ρ). But then the Hamilton cycle $S_0S_6S_9S_3S_2S_8S_1S_7S_5S_4S_0$ of $X_{\mathcal{S}}$ has a non-zero label $-t + b \neq 0$ (since $t = a \neq b$), and so it gives rise to a Hamilton cycle of X.

And second, suppose that $S_1S_9 \in E(X_{\mathcal{S}})$. Then whenever $t \neq a$ the Hamilton cycle given in (b) lifts to a Hamilton cycle of X. Thus we may assume that $t = a$. But then the Hamilton cycle $S_0S_6S_8S_2S_3S_9S_1S_7S_5S_4S_0$ of $X_{\mathcal{S}}$ has a non-zero label $b - a$, and so it gives rise to a Hamilton cycle of X.

If, however $a \neq d$ then $b = c$ and thus also $d = e = c = b$. As in the previous case in view of vertex-transitivity of X we get that Y_2 is a proper subgraph of $X_{\mathcal{S}}$. Also, if we consider all possibilities for edges of $X_{\mathcal{S}}$ lying outside the subgraph Y_2 and containing S_0 (in such a way as for S_1 in the previous case) we get that X has a Hamilton cycle also in this case.

Subcase 3.2.3. $X_{\mathcal{S}}$ contains Y_3.

Observe that every edge of Y_3 is contained in a Hamilton cycle and thus we may assume that all the edges of Y_3 are single edges in X_ρ. Further, since

$$S_0S_6S_9S_5S_4S_3S_2S_8S_7S_1S_0 \qquad S_0S_6S_8S_7S_1S_2S_3S_9S_5S_4S_0$$
$$S_0S_1S_2S_3S_9S_6S_8S_7S_5S_4S_0 \qquad S_0S_1S_7S_5S_9S_6S_8S_2S_3S_4S_0$$

are Hamilton cycles in Y_3, we can assume that they all carry label 0. Combining together the corresponding equations for the labels of arcs in Y_3 imply that all the edges in Y_3 carry label 0. It follows that Y_3 is a proper subgraph of X_S, implying that there must exist an arc e in X_S carrying a non-zero label. From symmetry reasons we may assume that either S_0, or S_1, or S_2 is an endvertex of this arc. In particular, the following cases need to be considered: $e = S_0S_5$, $e = S_0S_8$, $e = S_1S_8$, $e = S_1S_9$, $e = S_1S_5$, $e = S_1S_6$, $e = S_2S_7$, $e = S_2S_9$, $e = S_2S_5$, and $e = S_2S_6$. However, since we have the following:

- if $e = S_0S_5$, then $S_0S_5S_7S_1S_2S_8S_6S_9S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
- if $e = S_0S_8$, then $S_0S_8S_6S_9S_5S_7S_1S_2S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
- if $e = S_1S_8$, then $S_0S_6S_9S_5S_7S_8S_1S_2S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;

– if $e = S_1S_9$, then $S_0S_6S_8S_7S_5S_9S_1S_2S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
– if $e = S_1S_5$, then $S_0S_6S_9S_3S_2S_8S_7S_1S_5S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
– if $e = S_1S_6$, then $S_0S_1S_6S_9S_5S_7S_8S_2S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
– if $e = S_2S_9$, then $S_0S_6S_8S_7S_5S_4S_3S_9S_2S_1S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
– if $e = S_2S_5$, then $S_0S_1S_7S_8S_6S_9S_5S_2S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;

we can assume that $e = S_2S_7$. Since X is regular and edges in Y_3 are single edges there exists an edge f with endvertex S_1 that is not contained in Y_3. In particular, either $f = S_1S_5$, or $f = S_1S_6$, or $f = S_1S_8$, or $f = S_1S_9$. Assume first that $f \neq S_1S_6$. Then, in view of the first part of this paragraph, we can assume that the edge f carries label 0, and consequently the following hold:

– if $e = S_1S_5$, then $S_0S_6S_8S_7S_2S_1S_5S_9S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
– if $e = S_1S_8$, then $S_0S_6S_8S_1S_2S_7S_5S_9S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
– if $e = S_1S_9$, then $S_0S_6S_8S_2S_7S_5S_4S_3S_2S_1S_0$ is a Hamilton cycle of X_S carrying a non-zero label.

It follows that either X has a Hamilton path or $f = S_1S_6$. If, however, $f = S_1S_6$, then since $S_0S_1S_6S_9S_5S_7S_8S_2S_3S_4S_0$ is a Hamilton cycle of $X_{\mathcal{S}}$ we can assume that it has label 0, and consequently that f carries label 0. But then $S_0S_1S_6S_8S_2S_7S_5S_9S_3S_4S_0$ is a Hamilton cycle of $X_{\mathcal{S}}$ carrying a non-zero label, and thus we can conclude that X possesses a Hamilton path also in this case.

Subcase 3.2.4. $X_{\mathcal{S}}$ contains Y_4.

Observe that every edge of Y_4 is contained in a Hamilton cycle and thus we may assume that all the edges of Y_4 are single edges in X_ρ. Further, since

$$S_0S_1S_2S_8S_5S_6S_7S_9S_3S_4S_0 \qquad S_0S_6S_5S_8S_9S_7S_1S_2S_3S_4S_0$$
$$S_0S_1S_2S_3S_4S_5S_8S_9S_7S_6S_0 \qquad S_0S_6S_7S_1S_2S_3S_9S_8S_5S_4S_0$$
$$S_0S_6S_5S_4S_3S_2S_8S_9S_7S_1S_0$$

are Hamilton cycles in Y_4, we can assume that they all carry label 0. Combining together the corresponding equations for the labels of arcs in Y_4 imply that either X has a Hamilton cycle or all the edges in Y_4 carry label 0. In particular, we may assume that Y_4 is a proper subgraph of X_S, implying that there must exist an arc e in X_S with a non-zero label. From symmetry reasons we may assume that either S_0, or S_1, or S_2 is an endvertex of this arc. In particular, the following cases need to be considered: $e = S_0S_5$, $e = S_0S_8$, $e = S_1S_8$, $e = S_1S_9$, $e = S_1S_5$, $e = S_1S_6$, $e = S_2S_7$, $e = S_2S_9$, $e = S_2S_5$, and $e = S_2S_6$. However, since the following hold:

– if $e = S_0S_5$, then $S_0S_5S_4S_3S_9S_8S_2S_1S_7S_6S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
– if $e = S_0S_8$, then $S_0S_8S_9S_7S_1S_2S_3S_4S_5S_6S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
– if $e = S_1S_8$, then $S_0S_1S_8S_2S_3S_9S_7S_6S_5S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
– if $e = S_1S_9$, then $S_0S_1S_9S_7S_6S_5S_8S_2S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
– if $e = S_1S_5$, then $S_0S_1S_5S_6S_7S_9S_8S_2S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
– if $e = S_1S_6$, then $S_0S_6S_1S_7S_9S_3S_2S_8S_5S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
– if $e = S_2S_7$, then $S_0S_1S_2S_7S_6S_5S_8S_9S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
– if $e = S_2S_9$, then $S_0S_1S_7S_6S_5S_8S_9S_2S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
– if $e = S_2S_5$, then $S_0S_1S_7S_6S_5S_2S_8S_9S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label;
– if $e = S_2S_6$, then $S_0S_1S_7S_9S_8S_5S_6S_2S_3S_4S_0$ is a Hamilton cycle of X_S carrying a non-zero label.

We can conclude that X has a Hamilton cycle in this case.

Claim 4. *If N^B is primitive on B, then X contains a Hamilton path.*

Since N^B is primitive either $X\langle B\rangle$ is a connected graph or it is totally disconnected. In the former case Claim 1 applies, whereas, in the latter case Claim 2 applies.

Claim 5. *If N^B is imprimitive on B, then X contains a Hamilton path.*

Let $T = N^B$. Since T is a non-abelian simple group, it is quasiprimitive on B. Let Δ be the corresponding imprimitivity block system of T on B. Since T is simple the kernel of the action of T on $X\langle B\rangle_\Delta$ is trivial, and so, by Proposition 2.17, T is a transitive group of degree $|\Delta|$. It follows that T is isomorphic to a subgroup of $S_{|\Delta|}$. Observe that Δ cannot consist of blocks of size p. Namely, if this is the case, then $|\Delta| = 5$ and consequently $T \leq S_5$. But this is clearly impossible as $p > 7$ divides $|T|$ (since T is a group of degree $5p$). We therefore have that $\Delta = \{\Delta_i \mid i \in \mathbb{Z}_p\}$ consists of p blocks of size 5. Then $X\langle \Delta_i\rangle$, $i \in \mathbb{Z}_p$, is a vertex-transitive graph of order 5, and thus it is isomorphic to $5K_1$, C_5 or K_5. Observe also that the corresponding quotient action on $X\langle B\rangle_\Delta$ is primitive, implying that either $X\langle B\rangle_\Delta \cong K_p$ or $X\langle B\rangle_\Delta \cong pK_1$.

Suppose first that $X\langle \Delta_i\rangle \cong 5K_1$. If $X\langle B\rangle_\Delta \cong pK_1$, then $X\langle B\rangle \cong 5pK_1$, and, by Claim 2, X has a Hamilton path. We may therefore assume that $X\langle B\rangle_\Delta \cong K_p$. If $X\langle B\rangle$ is a connected graph, then by Claim 1, X has a Hamilton path. If, however, $X\langle B\rangle$ is a disconnected graph (but clearly not totally disconnected), then since $p > 5$ and since, by assumption the graphs induced on the blocks Δ_i, $i \in \mathbb{Z}_p$, are isomorphic to $5K_1$, the connected components of $X\langle B\rangle$ are of size p. However these connected components form an imprimitivity block system $\mathcal{D}$ of T on B consisting of blocks of size p, which in view of the argument given in the first paragraph of the proof of this claim is impossible.

Next, suppose that $X\langle \Delta_i\rangle \cong C_5$. If $X\langle B\rangle_\Delta \cong K_p$, then $X\langle B\rangle$ is a connected graph, and, by Claim 1, X has a Hamilton path. If, however, $X\langle B\rangle_\Delta \cong pK_1$, then $X\langle B\rangle$ is disconnected and $X\langle B\rangle \cong X\langle B'\rangle \cong pC_5$, and thus, by Claim 3, X has a Hamilton path.

Finally, suppose that $X\langle \Delta_i\rangle \cong K_5$. If $X\langle B\rangle$ is a connected graph, then by Claim 1, X has a Hamilton path. If, however, $X\langle B\rangle$ is disconnected, then $X\langle B\rangle \cong pK_5$, and clearly also $X\langle B'\rangle \cong pK_5$. The imprimitivity block system Δ on B gives rise to an imprimitivity block system of G on X, and in addition, the quotient graph with respect to this imprimitivity block system is a bipartite connected vertex-transitive graph of order $2p$. Let $V(X) = \{u_i^j \mid i \in \mathbb{Z}_{10}, j \in \mathbb{Z}_p\}$ such that the sets $\{u_i^j \mid j \in \mathbb{Z}_p\}$, $i \in \mathbb{Z}_{10}$ are orbits of ρ. Then, without loss of generality, we may assume that $B = \{u_i^j \mid i \in \{0, 1, 2, 3, 4\}, j \in \mathbb{Z}_p\}$, $B' = \{u_i^j \mid i \in \{5, 6, 7, 8, 9\}, j \in \mathbb{Z}_p\}$, and that $F_j = \{u_i^j \mid i \in \{0, 1, 2, 3, 4\}\}$ and $T_j = \{u_i^j \mid i \in \{5, 6, 7, 8, 9\}\}$, $j \in \mathbb{Z}_p$, are the connected components of $X\langle B\rangle$ and $X\langle B'\rangle$, respectively. Then $\mathcal{C} = \{F_j, T_j \mid j \in \mathbb{Z}_p\}$ is an imprimitivity block system of G on X with blocks of size 5. Since X is connected there must exist two vertices in F_0 that have neighbors in two different blocks of $\mathcal{C}$ lying in B'. Since the graph induced on F_0 is isomorphic to K_5, we may, without loss of generality, assume that u_0^0 is adjacent to u_5^0 and that u_1^0 is adjacent to u_k^j, where $j \neq 0$ and $k \in \{5, 6, 7, 8, 9\}$. Now one can, with the use of a $(2, p)$-semiregular automorphism of $X_{\mathcal{C}}$ (arising from the $(10, p)$-semiregular automorphism ρ of X), see that each of these two edges gives a perfect matching in $X_{\mathcal{C}}$. Moreover, since $j \neq 0$, we have that the union of these two perfect matchings is a Hamilton cycle of $X_{\mathcal{C}}$. Since $X\langle F_j\rangle \cong K_5$ and $X\langle T_j\rangle \cong K_5$ are Hamilton-connected we can clearly conclude that X has a Hamilton path. □

4. Quasiprimitive graphs

Throughout this section let X denote a connected quasiprimitive graph of order $10p$, $p \geq 7$ a prime. In [39] a complete characterization of quasiprimitive graphs of order pqr, where p, q and r are distinct primes, was given via the well-known generalized orbital graph construction relative to certain simple groups having an imprimitive permutation representation of degree pqr. All the possible group actions are given in Tables A and B in [39, p. 298–299]. For our purposes (we require that $pqr = 10p'$) only a handful of group actions needs to be considered. They are given in Table 1. Note that only row 16 of Table 1 corresponds to an infinite family of actions giving rise to quasiprimitive graphs of order $10p$. As for the other rows of Table 1, each case is investigated separately. More precisely, we consider all the possible generalized orbital graphs and study their structural properties (using program package Magma [5]) which allows us to easily find a Hamilton cycle in these graphs.

Let G be a group acting on the cosets of its subgroup H in a natural way. Following the terminology of [26] we say that the set $\mathcal{O}(G, H)$ of generalized orbital graphs (in short GOGs) of this action is a *minimal connected orbital graph set* for this action if each connected GOG corresponding to this action contains some graph of $\mathcal{O}(G, H)$ as a spanning subgraph. As we are only interested in whether a given

GOG contains a Hamilton path (or a Hamilton cycle) Proposition 2.5 implies that we can disregard the graphs from $\mathcal{O}(G, H)$ whose valencies are at least $[G : H]/3$. We let the remaining set of GOGs be the set $\mathcal{R}(G, H)$ of *relevant graphs* for this action. It is now clear that in order to show that each GOG corresponding to the above mentioned action of G contains a Hamilton path (Hamilton cycle) we only need to show that each GOG of $\mathcal{R}(G, H)$ has this property.

We now describe the method of obtaining $\mathcal{R}(G, H)$ for the action of row 2 of Table 1 in full detail. The other actions are dealt with in a similar way, so we only give the relevant graphs and leave the details to the reader. Each relevant graph X will be represented in a structural way given by some semiregular automorphism φ of X from which the existence of a Hamilton cycle will be clear. In the case when φ is $(10, p)$-semiregular its symbol (for the definition see Section 2) will be given.

Graphs corresponding to row 2 *of* Table 1: Note that these graphs are of order 110. In the action of PSL(2, 11) on the cosets of D_6, we get that D_6 has 21 nontrivial suborbits, nine of which are self-paired. Of the nine self-paired suborbits, one is of length 1 and two are of length 3, the others are of length 6. Of the twelve non-self-paired suborbits, 2 are of length 3, the others are of length 6. Denote these 21 nontrivial suborbits by U_i, $i \in \{1, 2, \ldots, 21\}$, where U_1 is of length 1, U_2 and U_3 are of length 3, the others are of length 6, $U_1, U_2, \ldots, U_9$ are the self-paired suborbits, and U_{2i} is paired with U_{2i+1} for $i \in \{5, 6, \ldots, 10\}$.

The unions $U_{2i} \cup U_{2i+1}$, where $i \in \{5, 6, \ldots, 10\}$, give rise to five nonisomorphic graphs, all of them are connected. Of these five graphs, three graphs admit a transitive group of automorphisms with a cyclic commutator subgroup of prime-power order, and thus, by Proposition 2.9, have a Hamilton cycle. The other two graphs are isomorphic to X_2 and X_3 of Table 2, respectively. Using an argument similar to the one used in the proof of Proposition 2.10, one can see that these two graphs both contain a Hamilton cycle.

For $i \in \{1, 2, 3, \ldots, 9\}$ the graphs arising from the suborbits U_i, $i \in \{6, 7, 8, 9\}$, are all connected. Moreover, the graphs arising from the suborbits U_7 and U_8 admit a transitive group of automorphisms with a cyclic commutator subgroup of prime-power order, and thus Proposition 2.9 implies that these graphs contain a Hamilton cycle. The graph arising from the suborbit U_6 is isomorphic to the graph arising from the suborbit U_9, and is isomorphic to the graph X_1 in Table 2. Proposition 2.10 implies that X_1 contains a Hamilton cycle.

The graphs arising from the suborbits U_i, $i \in \{1, 2, 3, 4, 5\}$, are disconnected, whereas the graphs arising from $U_1 \cup U_i$, $i \in \{2, 3, 4, 5\}$, are connected and give rise to two nonisomorphic graphs X_4 and X_5 in Table 2. Proposition 2.10 implies that both graphs contain a Hamilton cycle. The graph X_4 is also given in Fig. 2.

Finally, the unions $U_i \cup U_j$, where $i, j \in \{2, 3, 4, 5\}$, give rise to three nonisomorphic connected graphs. Two of them admit a transitive group of automorphisms with a cyclic commutator subgroup of prime-power order, and thus, by Proposition 2.9, have a Hamilton cycle. The third graph is isomorphic to the graph X_6 in Table 2. Proposition 2.10 implies that this graph has a Hamilton cycle.

We have now clearly considered all the relevant graphs $\mathcal{R}(\mathrm{PSL}(2, 11), D_6)$, and we can conclude that each connected GOG arising from the action of PSL(2, 11) on the cosets of D_6 contains a Hamilton cycle.

Graphs corresponding to row 1 *of* Table 1: The relevant graphs are given in Table 3, and so it is clear that each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 3 *of* Table 1: The relevant graphs are given in Table 4, and, by Proposition 2.10, each of them contains a Hamilton cycle.

Graphs corresponding to row 4 *of* Table 1: It turns out that $\mathcal{R}(G, H) = \emptyset$ in this case, and so each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 5 *of* Table 1: It turns out that $\mathcal{R}(G, H) = \emptyset$ in this case, and so each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 6 *of* Table 1: There are four connected relevant graphs. They all admit a transitive group of automorphisms with a cyclic commutator subgroup of prime-power order. Thus Proposition 2.9 implies that these graphs have a Hamilton cycle.

Table 2
Relevant graphs corresponding to the action of row 2 of Table 1.

	X_1	X_2	X_3	X_4	X_5	X_6
p	11	11	11	11	11	11
$\|V(X_i)\|$	110	110	110	110	110	110
Val	6	12	10	4	7	9
$R_{0,0}$	$\emptyset$	$\pm4, \pm5$	±4	$\emptyset$	$\emptyset$	$\emptyset$
$R_{1,1}$	$\emptyset$	$\pm2, \pm5$	±4	$\emptyset$	$\emptyset$	$\emptyset$
$R_{2,2}$	±3	$\pm1, \pm3$	±1	$\emptyset$	$\emptyset$	$\emptyset$
$R_{3,3}$	$\emptyset$	$\emptyset$	±1	$\emptyset$	$\emptyset$	$\emptyset$
$R_{4,4}$	$\emptyset$	$\emptyset$	±5	$\emptyset$	$\emptyset$	$\emptyset$
$R_{5,5}$	±2	$\emptyset$	±5	$\emptyset$	$\emptyset$	$\emptyset$
$R_{6,6}$	±1	$\emptyset$	±3	$\emptyset$	$\emptyset$	$\emptyset$
$R_{7,7}$	±5	$\pm1, \pm4$	±2	$\emptyset$	$\emptyset$	$\emptyset$
$R_{8,8}$	±4	$\pm2, \pm3$	±3	$\emptyset$	$\emptyset$	$\emptyset$
$R_{9,9}$	$\emptyset$	$\pm4, \pm5$	±2	$\emptyset$	$\emptyset$	$\emptyset$
$R_{0,1}$	0	0, 7	0, 8	0	0	0
$R_{0,2}$	0, 3	0, 9	0, 10	0	0, 10	0, 8
$R_{0,3}$	$\emptyset$	0	0, 10	0	0	0, 3
$R_{0,4}$	0	0, 6	0, 7	0	0	0
$R_{0,5}$	0	0, 7	0, 4	$\emptyset$	0	0, 6
$R_{0,6}$	0	0	$\emptyset$	$\emptyset$	0	0
$R_{0,7}$	$\emptyset$	0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$R_{0,8}$	$\emptyset$	0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$R_{0,9}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$R_{1,2}$	$\emptyset$	1	6, 7	$\emptyset$	0	8
$R_{1,3}$	$\emptyset$	10	6, 7	7	0	$\emptyset$
$R_{1,4}$	$\emptyset$	$\emptyset$	3, 7	$\emptyset$	0	0, 8
$R_{1,5}$	$\emptyset$	6	0, 7	0	$\emptyset$	0, 5
$R_{1,6}$	0	6, 10	$\emptyset$	0	$\emptyset$	$\emptyset$
$R_{1,7}$	0	6	$\emptyset$	$\emptyset$	0	0, 9
$R_{1,8}$	0, 4	$\emptyset$	$\emptyset$	$\emptyset$	0, 3	0
$R_{1,9}$	0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$R_{2,3}$	0	3, 5	1, 10	$\emptyset$	0	6
$R_{2,4}$	$\emptyset$	3	$\emptyset$	1	$\emptyset$	$\emptyset$
$R_{2,5}$	$\emptyset$	3	$\emptyset$	$\emptyset$	0	$\emptyset$
$R_{2,6}$	$\emptyset$	$\emptyset$	0, 8	1	$\emptyset$	3
$R_{2,7}$	$\emptyset$	$\emptyset$	$\emptyset$	0	0	3, 10
$R_{2,8}$	$\emptyset$	$\emptyset$	0, 8	$\emptyset$	$\emptyset$	3, 7
$R_{2,9}$	7	0	$\emptyset$	$\emptyset$	0	$\emptyset$
$R_{3,4}$	7	1, 3	$\emptyset$	$\emptyset$	$\emptyset$	8, 9
$R_{3,5}$	$\emptyset$	3	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$R_{3,6}$	2, 3	5, 8	0, 8	$\emptyset$	0, 6	4, 8
$R_{3,7}$	$\emptyset$	8	$\emptyset$	10	$\emptyset$	$\emptyset$
$R_{3,8}$	5	$\emptyset$	0, 8	0	0	8
$R_{3,9}$	$\emptyset$	0, 10	$\emptyset$	$\emptyset$	0	0
$R_{4,5}$	0	$\emptyset$	9, 10	3	0, 9	0
$R_{4,6}$	$\emptyset$	6	$\emptyset$	$\emptyset$	0	$\emptyset$
$R_{4,7}$	2, 8	6, 9	0, 6	$\emptyset$	0	$\emptyset$
$R_{4,8}$	10	2, 7	$\emptyset$	$\emptyset$	0	0, 9
$R_{4,9}$	9	5	0, 6	0	$\emptyset$	3
$R_{5,6}$	$\emptyset$	9, 10	$\emptyset$	$\emptyset$	0	0
$R_{5,7}$	$\emptyset$	$\emptyset$	2, 7	10	0	0
$R_{5,8}$	$\emptyset$	2, 7	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$R_{5,9}$	5	0, 2	2, 7	0	0	2, 3
$R_{6,7}$	$\emptyset$	2, 10	2, 4	$\emptyset$	$\emptyset$	0, 6
$R_{6,8}$	$\emptyset$	5	3, 8	0	0	$\emptyset$
$R_{6,9}$	$\emptyset$	9	2, 4	2	0	3, 4
$R_{7,8}$	$\emptyset$	2	7, 9	7	0	0
$R_{7,9}$	0	$\emptyset$	2, 9	$\emptyset$	0, 7	3
$R_{8,9}$	$\emptyset$	0	2, 4	0	0	1, 3

Table 3
Relevant graphs corresponding to the action of row 1 of Table 1.

	X_1	X_2
p	7	7
$\lvert V(X_i)\rvert$	70	70
Val	6	18
$R_{0,0}$	$\emptyset$	$\emptyset$
$R_{1,1}$	$\emptyset$	± 3
$R_{2,2}$	$\emptyset$	± 2
$R_{3,3}$	$\emptyset$	± 1
$R_{4,4}$	± 3	± 1
$R_{5,5}$	$\emptyset$	± 1
$R_{6,6}$	$\emptyset$	± 3
$R_{7,7}$	$\emptyset$	± 2
$R_{8,8}$	± 1	± 2
$R_{9,9}$	± 2	± 3
$R_{0,1}$	0	0
$R_{0,2}$	0	0
$R_{0,3}$	0	0
$R_{0,4}$	0, 3	0, 3
$R_{0,5}$	0	0
$R_{0,6}$	$\emptyset$	0
$R_{0,7}$	$\emptyset$	0
$R_{0,8}$	$\emptyset$	0, 1, 4, 5
$R_{0,9}$	$\emptyset$	0
$R_{1,2}$	2	$\emptyset$
$R_{1,3}$	0	± 1
$R_{1,4}$	$\emptyset$	1, 3, 5
$R_{1,5}$	0	$\emptyset$
$R_{1,6}$	0	3, 5
$R_{1,7}$	0	4, 5, 6
$R_{1,8}$	$\emptyset$	2, 3
$R_{1,9}$	$\emptyset$	1, 2, 3
$R_{2,3}$	$\emptyset$	1, 4
$R_{2,4}$	0, 3	2, 3, 6
$R_{2,5}$	$\emptyset$	$\emptyset$
$R_{2,6}$	5	1, 2
$R_{2,7}$	5	1, 4, 5
$R_{2,8}$	$\emptyset$	3, 4
$R_{2,9}$	$\emptyset$	4, 5, 6
$R_{3,4}$	$\emptyset$	1, 5
$R_{3,5}$	0	0, 4
$R_{3,6}$	$\emptyset$	$\emptyset$
$R_{3,7}$	3	1, 4
$R_{3,8}$	0, 1	$\emptyset$
$R_{3,9}$	$\emptyset$	4, 6
$R_{4,5}$	$\emptyset$	1, 3, 5
$R_{4,6}$	$\emptyset$	1, 3
$R_{4,7}$	$\emptyset$	$\emptyset$
$R_{4,8}$	$\emptyset$	3, 6
$R_{4,9}$	$\emptyset$	$\emptyset$
$R_{5,6}$	6	1, 3
$R_{5,7}$	$\emptyset$	2, 3, 6
$R_{5,8}$	$\emptyset$	2, 6
$R_{5,9}$	0, 2	1, 3, 5
$R_{6,7}$	0	4, 5
$R_{6,8}$	$\emptyset$	$\emptyset$
$R_{6,9}$	1, 3	1, 6
$R_{7,8}$	4, 5	1, 2
$R_{7,9}$	$\emptyset$	$\emptyset$
$R_{8,9}$	$\emptyset$	4, 5

Table 4
Relevant graphs corresponding to the action of row 3 of Table 1.

	X_1	X_2
p	11	11
$\|V(X_i)\|$	110	110
Val	12	6
$R_{0,0}$	± 3	± 1
$R_{1,1}$	± 1	$\emptyset$
$R_{2,2}$	± 2	$\emptyset$
$R_{3,3}$	± 2	± 2
$R_{4,4}$	± 1	$\emptyset$
$R_{5,5}$	± 3	± 5
$R_{6,6}$	± 4	$\emptyset$
$R_{7,7}$	± 4	± 4
$R_{8,8}$	± 5	$\emptyset$
$R_{9,9}$	± 4	± 3
$R_{0,1}$	0, 8	0, 1
$R_{0,2}$	0, 2	0, 10
$R_{0,3}$	0, 2	$\emptyset$
$R_{0,4}$	0, 8	$\emptyset$
$R_{0,5}$	0, 5	$\emptyset$
$R_{0,6}$	$\emptyset$	$\emptyset$
$R_{0,7}$	$\emptyset$	$\emptyset$
$R_{0,8}$	$\emptyset$	$\emptyset$
$R_{0,9}$	$\emptyset$	$\emptyset$
$R_{1,2}$	$\emptyset$	10
$R_{1,3}$	$\emptyset$	0, 2
$R_{1,4}$	± 1	0
$R_{1,5}$	0, 8	$\emptyset$
$R_{1,6}$	0, 10	$\emptyset$
$R_{1,7}$	0, 10	$\emptyset$
$R_{1,8}$	$\emptyset$	$\emptyset$
$R_{1,9}$	$\emptyset$	$\emptyset$
$R_{2,3}$	± 2	$\emptyset$
$R_{2,4}$	$\emptyset$	$\emptyset$
$R_{2,5}$	6, 8	0, 6
$R_{2,6}$	$\emptyset$	0
$R_{2,7}$	$\emptyset$	$\emptyset$
$R_{2,8}$	0, 5	$\emptyset$
$R_{2,9}$	0, 6	$\emptyset$
$R_{3,4}$	$\emptyset$	0, 9
$R_{3,5}$	6, 8	$\emptyset$
$R_{3,6}$	$\emptyset$	$\emptyset$
$R_{3,7}$	$\emptyset$	$\emptyset$
$R_{3,8}$	0, 5	$\emptyset$
$R_{3,9}$	0, 6	$\emptyset$
$R_{4,5}$	0, 8	$\emptyset$
$R_{4,6}$	0, 10	$\emptyset$
$R_{4,7}$	0, 10	0, 4
$R_{4,8}$	$\emptyset$	0
$R_{4,9}$	$\emptyset$	$\emptyset$
$R_{5,6}$	$\emptyset$	0, 5
$R_{5,7}$	$\emptyset$	$\emptyset$
$R_{5,8}$	$\emptyset$	$\emptyset$
$R_{5,9}$	0, 2	$\emptyset$
$R_{6,7}$	4, 7	$\emptyset$
$R_{6,8}$	± 2	1
$R_{6,9}$	4, 8	0, 8
$R_{7,8}$	± 2	0, 7
$R_{7,9}$	4, 8	$\emptyset$
$R_{8,9}$	0, 1	7, 10

Graphs corresponding to row 7 *of* Table 1: It turns out that $\mathcal{R}(G, H) = \emptyset$ in this case, and so each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 8 *of* Table 1: The relevant graphs are given in Table 5. By Proposition 2.10, each of these graphs contains a Hamilton cycle.

Graphs corresponding to row 9 *of* Table 1: The relevant graphs are given in Table 6. By Proposition 2.10, each of these graphs contains a Hamilton cycle.

Graphs corresponding to row 10 *of* Table 1: It turns out that $\mathcal{R}(G, H) = \emptyset$ in this case, and so each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 11 *of* Table 1: There is only one connected relevant graph. It admits a transitive group of automorphisms with a cyclic commutator subgroup of prime-power order. By Proposition 2.9, this graph thus has a Hamilton cycle.

Graphs corresponding to row 12 *of* Table 1: The relevant graphs are given in Table 9. By Proposition 2.10, each of these graphs contains a Hamilton cycle.

Graphs corresponding to row 13 *of* Table 1: It turns out that $\mathcal{R}(G, H) = \emptyset$ in this case, and so each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 14 *of* Table 1: The relevant graphs are given in Table 7. By Proposition 2.10, each of these graphs contains a Hamilton cycle.

Graphs corresponding to row 15 *of* Table 1: There are two connected relevant graphs. They both admit a transitive group of automorphisms with a cyclic commutator subgroup of prime-power order. By Proposition 2.9, these graphs thus have a Hamilton cycle.

In view of the fact that the connected quasiprimitive graphs of orders $4p$, $2p^2$, and $6p$ (except for the truncation of the Petersen graph) contain a Hamilton cycle (see [24,25,32]), the results of this section imply that the following proposition holds.

Proposition 4.1. *Let X be a connected quasiprimitive graph of order $10p, p$ a prime, which is not isomorphic to a quasiprimitive graph arising from the action of* $\mathrm{PSL}(2, k)$ *on cosets of* $\mathbb{Z}_k \rtimes \mathbb{Z}_{(k-1)/10}$. *Then X is the truncation of the Petersen graph or X is Hamiltonian.*

5. Primitive graphs

Throughout this section let X denote a primitive graph of order $10p$, p a prime. In [16] the complete characterization of the primitive graphs of order $2pq$, where p and q are distinct odd primes, was given. Extracting the information about graphs of order $10p$ we find that the only primitive graphs of order $10p$, p a prime, are the ones arising from the actions given in Table 10. Below we show that each of the corresponding graphs has a Hamilton cycle. We let the GOGs and the relevant graphs corresponding to some action be defined as in Section 4.

Graphs corresponding to row 1 of Table 10: It turns out that $\mathcal{R}(G, H) = \emptyset$ in this case, and so each GOG arising from this action contains a Hamilton cycle.

Graphs corresponding to row 2 of Table 10: The relevant graphs are given in Table 8, and so it is clear that each GOG arising from this action contains a Hamilton cycle.

The results of this section imply that the following proposition holds.

Proposition 5.1. *A primitive graph of order* $10p$, *p a prime, contains a Hamilton cycle.*

6. The proof of the main theorem

Proof of Theorem 1.1. If X is not genuinely imprimitive, then either Proposition 4.1 or Proposition 5.1 applies. If, however, X is genuinely imprimitive, then in view of the fact that the connected vertex-transitive graphs of orders $4p$, $6p$ and $2p^2$ contain a Hamilton path (see [24,25,32]), we may assume that $p > 7$. Now apply one of Lemmas 3.1 and 3.3–3.7, depending on the size of the corresponding blocks.

Table 5
Relevant graphs corresponding to the action of row 8 of Table 1.

	X_1	X_2	X_3
p	7	7	7
$\|V(X_i)\|$	70	70	70
Val	8	12	12
$R_{0,0}$	± 3	± 1	$\emptyset$
$R_{1,1}$	± 3	± 3	$\emptyset$
$R_{2,2}$	$\emptyset$	± 2	$\emptyset$
$R_{3,3}$	± 2	± 1	$\emptyset$
$R_{4,4}$	$\emptyset$	± 1	$\emptyset$
$R_{5,5}$	$\emptyset$	± 1	$\emptyset$
$R_{6,6}$	± 2	± 3	$\emptyset$
$R_{7,7}$	± 2	± 2	$\emptyset$
$R_{8,8}$	± 3	± 2	$\emptyset$
$R_{9,9}$	± 1	± 3	$\emptyset$
$R_{0,1}$	0	0, 6	0, 1
$R_{0,2}$	0	0, 6	0
$R_{0,3}$	0, 6	0, 2	0, 2
$R_{0,4}$	0	0, 1	0, 6
$R_{0,5}$	0	0	0, 6
$R_{0,6}$	$\emptyset$	0	0
$R_{0,7}$	$\emptyset$	$\emptyset$	0, 5
$R_{0,8}$	$\emptyset$	$\emptyset$	$\emptyset$
$R_{0,9}$	$\emptyset$	$\emptyset$	$\emptyset$
$R_{1,2}$	5	0	5
$R_{1,3}$	3	$\emptyset$	2, 6
$R_{1,4}$	$\emptyset$	1	0, 3
$R_{1,5}$	5	2, 5	2, 5
$R_{1,6}$	0	3, 6	1
$R_{1,7}$	0	$\emptyset$	$\emptyset$
$R_{1,8}$	0	$\emptyset$	0, 6
$R_{1,9}$	0	$\emptyset$	$\emptyset$
$R_{2,3}$	3	4	5
$R_{2,4}$	2	1, 5, 6	1, 5, 6
$R_{2,5}$	$\emptyset$	$\emptyset$	$\emptyset$
$R_{2,6}$	1	$\emptyset$	$\emptyset$
$R_{2,7}$	1	1, 6	0, 1
$R_{2,8}$	5	0, 4	2, 6
$R_{2,9}$	5	$\emptyset$	0, 2
$R_{3,4}$	4	2	3, 5
$R_{3,5}$	4	0, 5	1, 6
$R_{3,6}$	$\emptyset$	0, 5	3
$R_{3,7}$	$\emptyset$	$\emptyset$	$\emptyset$
$R_{3,8}$	$\emptyset$	0, 3	$\emptyset$
$R_{3,9}$	$\emptyset$	$\emptyset$	2, 5
$R_{4,5}$	5	$\emptyset$	$\emptyset$
$R_{4,6}$	0	$\emptyset$	$\emptyset$
$R_{4,7}$	0	1, 3	1
$R_{4,8}$	0	1, 5	4
$R_{4,9}$	0	0	6
$R_{5,6}$	1	0, 2, 6	0, 2, 6
$R_{5,7}$	1	5	5
$R_{5,8}$	5	4	4
$R_{5,9}$	5	0, 1	4
$R_{6,7}$	± 2	4	4, 5
$R_{6,8}$	$\emptyset$	1	1, 5
$R_{6,9}$	$\emptyset$	4, 5	0, 2
$R_{7,8}$	$\emptyset$	$\emptyset$	4, 5
$R_{7,9}$	$\emptyset$	4, 6	1, 3
$R_{8,9}$	1, 6	3, 4	3, 6

Table 6
Relevant graphs corresponding to the action of row 9 of Table 1.

	X_1	X_2
p	7	7
$\|V(X_i)\|$	70	70
Val	16	16
$R_{0,0}$	$\emptyset$	$\emptyset$
$R_{1,1}$	± 2	± 2
$R_{2,2}$	± 1	± 3
$R_{3,3}$	± 1	$\emptyset$
$R_{4,4}$	± 3	$\emptyset$
$R_{5,5}$	$\emptyset$	± 2
$R_{6,6}$	± 2	$\emptyset$
$R_{7,7}$	$\emptyset$	± 1
$R_{8,8}$	± 3	± 3
$R_{9,9}$	$\emptyset$	$\emptyset$
$R_{0,1}$	0	0
$R_{0,2}$	0, 1, 6	0, 1, 6
$R_{0,3}$	0	0, 2, 5
$R_{0,4}$	0, 1, 4	0, 2, 6
$R_{0,5}$	0, 4, 6	0, 3, 6
$R_{0,6}$	0, 2, 4	0
$R_{0,7}$	0	0
$R_{0,8}$	0	0
$R_{0,9}$	$\emptyset$	$\emptyset$
$R_{1,2}$	$\emptyset$	2, 3
$R_{1,3}$	0, 5	3, 6
$R_{1,4}$	$\emptyset$	1, 4, 5
$R_{1,5}$	4	4, 5
$R_{1,6}$	2, 4	1
$R_{1,7}$	0, 2, 4	$\emptyset$
$R_{1,8}$	0, 4	$\emptyset$
$R_{1,9}$	0, 2, 4	1, 2, 4
$R_{2,3}$	0, 6	$\emptyset$
$R_{2,4}$	1, 2	4
$R_{2,5}$	4, 5, 6	$\emptyset$
$R_{2,6}$	1, 6	2, 3, 4
$R_{2,7}$	1	2, 4
$R_{2,8}$	$\emptyset$	3, 5
$R_{2,9}$	4	3
$R_{3,4}$	$\emptyset$	3
$R_{3,5}$	6	$\emptyset$
$R_{3,6}$	$\emptyset$	1, 4, 6
$R_{3,7}$	0, 1, 2	3, 6
$R_{3,8}$	0, 6	1, 3
$R_{3,9}$	4, 5, 6	0
$R_{4,5}$	2, 3, 6	2
$R_{4,6}$	0, 4	$\emptyset$
$R_{4,7}$	0	2, 4, 6
$R_{4,8}$	3, 6	2, 3, 4
$R_{4,9}$	2	1
$R_{5,6}$	0, 2, 5	2, 5, 6
$R_{5,7}$	$\emptyset$	3, 6
$R_{5,8}$	0	5, 6
$R_{5,9}$	0	5
$R_{6,7}$	0	3
$R_{6,8}$	$\emptyset$	5
$R_{6,9}$	5	2, 4, 5
$R_{7,8}$	0, 3, 6	$\emptyset$
$R_{7,9}$	2, 3, 5	0, 3, 5
$R_{8,9}$	0, 3, 6	1, 2, 3

Table 7
Relevant graphs corresponding to the action of row 14 of Table 1.

	X_1
p	11
$\|V(X_i)\|$	110
Val	18
$R_{0,0}$	$\pm$ 4
$R_{1,1}$	$\pm$ 5
$R_{2,2}$	$\pm$ 3
$R_{3,3}$	$\pm$ 2
$R_{4,4}$	$\pm$ 1
$R_{5,5}$	$\pm$ 2
$R_{6,6}$	$\pm$ 1
$R_{7,7}$	$\pm$ 5
$R_{8,8}$	$\pm$ 4
$R_{9,9}$	$\pm$ 2
$R_{0,1}$	0
$R_{0,2}$	0
$R_{0,3}$	02, 3, 5
$R_{0,4}$	0, 34, 10
$R_{0,5}$	0, 4
$R_{0,6}$	0
$R_{0,7}$	0
$R_{0,8}$	0
$R_{0,9}$	$\emptyset$
$R_{1,2}$	2, 3, 7, 9
$R_{1,3}$	7
$R_{1,4}$	3, 8
$R_{1,5}$	1
$R_{1,6}$	1, 2, 9, 10
$R_{1,7}$	1
$R_{1,8}$	0
$R_{1,9}$	0
$R_{2,3}$	10
$R_{2,4}$	5
$R_{2,5}$	5
$R_{2,6}$	3
$R_{2,7}$	0, 8
$R_{2,8}$	9
$R_{2,9}$	7
$R_{3,4}$	0
$R_{3,5}$	1
$R_{3,6}$	0, 2, 6, 8
$R_{3,7}$	5
$R_{3,8}$	3, 5
$R_{3,9}$	8
$R_{4,5}$	2, 5, 8, 9
$R_{4,6}$	0
$R_{4,7}$	6
$R_{4,8}$	0, 5, 9, 10
$R_{4,9}$	3
$R_{5,6}$	9
$R_{5,7}$	7
$R_{5,8}$	2
$R_{5,9}$	0, 1, 2, 9
$R_{6,7}$	3
$R_{6,8}$	8
$R_{6,9}$	9, 10
$R_{7,8}$	1, 3, 7, 10
$R_{7,9}$	3, 4, 6, 9
$R_{8,9}$	2

Table 8
Relevant graphs corresponding to the action of row 2 of Table 10.

	X_1
p	19
$\|V(X_i)\|$	190
Val	36
$R_{0,0}$	± 5
$R_{1,1}$	± 9
$R_{2,2}$	± 1
$R_{3,3}$	± 3
$R_{4,4}$	± 8
$R_{5,5}$	± 7
$R_{6,6}$	$\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8, \pm 9$
$R_{7,7}$	± 2
$R_{8,8}$	± 6
$R_{9,9}$	± 4
$R_{0,1}$	7, 12, 13, 18
$R_{0,2}$	0, 1, 5, 6
$R_{0,3}$	0, 11, 14, 16
$R_{0,4}$	0, 3, 8, 14
$R_{0,5}$	0,5, 12, 17
$R_{0,6}$	0, 14
$R_{0,7}$	0, 3, 5, 17
$R_{0,8}$	0, 5, 6, 11
$R_{0,9}$	0, 4, 14, 18
$R_{1,2}$	5, 6, 14, 15
$R_{1,3}$	0, 6, 9, 16
$R_{1,4}$	0, 8, 9, 17
$R_{1,5}$	5, 7, 14, 17
$R_{1,6}$	0, 9
$R_{1,7}$	3, 5, 12, 14
$R_{1,8}$	1, 5, 11, 14
$R_{1,9}$	0, 4, 9, 13
$R_{2,3}$	10, 11, 13, 14
$R_{2,4}$	2, 3, 13, 14
$R_{2,5}$	50, 11, 12, 18
$R_{2,6}$	13, 14
$R_{2,7}$	0, 16, 17, 18
$R_{2,8}$	0, 5, 6, 18
$R_{2,9}$	13, 14, 17, 18
$R_{3,4}$	0, 3, 8, 11
$R_{3,5}$	1, 5, 8, 17
$R_{3,6}$	0, 3
$R_{3,7}$	3, 5, 6, 8
$R_{3,8}$	5, 8, 11, 14
$R_{3,9}$	0, 3, 4, 7
$R_{4,5}$	5, 9, 16, 17
$R_{4,6}$	0, 11
$R_{4,7}$	3, 5, 14, 16
$R_{4,8}$	3, 5, 11, 16
$R_{4,9}$	0, 4, 11, 15
$R_{5,6}$	2, 14
$R_{5,7}$	0, 5, 7, 17
$R_{5,8}$	0, 6, 7, 13
$R_{5,9}$	2, 6, 14, 18
$R_{6,7}$	3, 5
$R_{6,8}$	5, 11
$R_{6,9}$	0, 4
$R_{7,8}$	0, 2, 6, 8
$R_{7,9}$	1, 14, 16, 18
$R_{8,9}$	8, 12, 14, 18

Table 9
Relevant graphs corresponding to the action of row 12 of Table 1.

	X_1	X_2	X_3	X_4	X_5
p	13	13	13	13	13
$\lvert V(X_i)\rvert$	130	130	130	130	130
Val	12	20	30	30	24
$R_{0,0}$	∅	±1	$\pm1, \pm5$	±4	∅
$R_{1,1}$	$\pm2, \pm6$	$\pm2, \pm4$	±4	±5	∅
$R_{2,2}$	$\pm3, \pm4$	±5	$\pm1, \pm5$	±6	∅
$R_{3,3}$	∅	±1	±3	±1	∅
$R_{4,4}$	±5	$\pm3, \pm6$	±2	±1	∅
$R_{5,5}$	±1	$\pm2, \pm4$	±2	±6	∅
$R_{6,6}$	∅	±1	±6	±5	$\pm3, \pm5, \pm6$
$R_{7,7}$	∅	±5	±3	±4	$\pm1, \pm2, \pm4$
$R_{8,8}$	∅	$\pm3, \pm6$	∅	$\pm2, \pm3$	$\pm3, \pm5, \pm6$
$R_{9,9}$	∅	∅	±4	$\pm2, \pm3$	$\pm1, \pm2, \pm4$
$R_{0,1}$	0	0, 2	0, 2, 6, 8, 9, 12	0, 4, 5, 8	0, 3
$R_{0,2}$	0	0, 3, 8	0, 8, 10, 11	0, 4, 7, 10	0, 1, 8, 9
$R_{0,3}$	0, 9, 11	0, 1, 2	0, 10	0, 7, 9	0, 12
$R_{0,4}$	0, 11	0, 3	0, 11	0, 4, 6	0, 2, 10, 12
$R_{0,5}$	0	0, 11	0, 11	0, 2, 4, 11	0, 6, 7, 12
$R_{0,6}$	0	0, 11, 12	0, 3, 4, 7, 9, 11	∅	0
$R_{0,7}$	0, 9	0, 5, 10	0, 3	0, 2	0, 2, 6
$R_{0,8}$	0	0, 10	0	0, 12	0
$R_{0,9}$	∅	∅	0	0, 1	0, 2, 9
$R_{1,2}$	∅	3, 7, 11	0	2, 6, 12	2, 4
$R_{1,3}$	2	0	6, 7, 12	0, 3, 7, 8, 9	3, 6, 10, 12
$R_{1,4}$	∅	∅	1, 6, 9, 12	∅	2, 7
$R_{1,5}$	3, 11	0, 2, 7, 9	2, 4, 7, 10	3, 7, 10	1, 2
$R_{1,6}$	4	11	∅	8, 12	0, 6, 8
$R_{1,7}$	∅	0, 4, 8	6, 11, 12	2, 7, 11, 12	2, 11, 12
$R_{1,8}$	6	∅	6, 7, 9, 11, 12	2, 4, 5, 7, 8, 10	2, 4, 10
$R_{1,9}$	0, 1	0, 11	6, 12	∅	0, 1, 10
$R_{2,3}$	8	6	1, 4	0, 1, 2, 7	4, 12
$R_{2,4}$	7, 8	0, 6	8, 10	2, 7, 8, 9	1, 2, 6, 10
$R_{2,5}$	∅	0, 4, 8	6, 8	4, 7	1, 9, 11, 12
$R_{2,6}$	9	4	6	2, 6, 12	1, 4, 10
$R_{2,7}$	4, 12	2, 5	1, 4	0, 2, 9, 11	7
$R_{2,8}$	7	10	2, 3, 6, 8, 10, 12	3, 8	0, 3, 7
$R_{2,9}$	∅	0, 5, 10	0, 1, 4, 5, 7, 11	∅	12
$R_{3,4}$	7, 9	0, 1, 7	3, 4, 8, 12	0, 6	1, 12
$R_{3,5}$	12	11	0, 5, 6, 12	0, 6, 7, 8	2, 5
$R_{3,6}$	8	10, 12	8, 9, 10, 12	∅	4, 9, 11
$R_{3,7}$	8, 12	4	7, 9	0, 4, 11	6, 9, 10
$R_{3,8}$	8	4, 10, 11	6, 8, 9, 10	1, 3, 4, 5, 6, 8	3, 5, 10
$R_{3,9}$	∅	0, 11, 12	5, 6, 11	5	6, 7, 10
$R_{4,5}$	∅	∅	4, 7	2, 7, 8, 9	0, 7, 8, 12
$R_{4,6}$	4	∅	7, 8, 12	0, 4, 7, 11, 12	0, 4, 10
$R_{4,7}$	∅	10	1, 2, 7, 8	2, 4, 11	11
$R_{4,8}$	10	0, 3, 4, 7	3, 4, 12	8	1, 4, 10
$R_{4,9}$	1, 4	0, 10	1, 4, 6, 11	5, 7, 8, 9, 10	5
$R_{5,6}$	2, 12	0	1, 2, 10	5, 9, 12	2
$R_{5,7}$	∅	2, 6, 10	1, 5, 6, 10	2, 5, 8, 11	0, 4, 6
$R_{5,8}$	7, 10	∅	3, 11, 12	0, 8	5
$R_{5,9}$	∅	0, 2	1, 4, 9, 11	3, 11	3, 5, 9
$R_{6,7}$	∅	6	1, 2, 3, 5	0, 4, 9, 12	6, 11
$R_{6,8}$	1, 4, 7	0, 6, 12	1, 5	9	∅
$R_{6,9}$	2, 8	0, 1, 2	0, 1, 3, 5, 6	1, 2, 4, 5, 7, 12	1, 2
$R_{7,8}$	∅	0	3, 4, 5, 7	5, 6	0, 12
$R_{7,9}$	0, 4, 5, 12	0, 3, 8	2, 7, 8	5, 6	∅
$R_{8,9}$	2, 9	0, 3	∅	0, 1, 3, 11	4, 9

Table 10
Primes p for which there exists a graph X on $10p$ vertices such that $\mathrm{Aut}(X)$ and all vertex-transitive subgroups of $\mathrm{Aut}(X)$ act primitively on X.

Row	p	Action of $\mathrm{Aut}(X)$
1	13	$\mathrm{PSL}(4,3)$ on cosets of P_2
2	19	S_{20} on pairs

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